Lecture 24

Extrema under Constraints

So far, we have minimized/maximized functions $f(x_1, x_2)$ on their domains.

Sometimes, we need to find extrema of $f(x_1, x_2)$ under some additional constraint.

E.g. We want to find the maximum/minimum of

$$f(x_1, x_2) = x_1 \cdot x_2 + 2x_1^2 x_2^2$$

under the constraint

$$(x_1 - 1)^2 + 2(x_2 - 2)^2 = 2.$$

In general, we want to find the extrema of the function $f(x_1, x_2)$ under the constraint

$$\boxed{g(x_1, x_2) = 0}$$

Theorem (Lagrange): Assume that f, g have continuous first-order partial derivatives and $f(x_1, x_2)$ has an extremum at $(\bar{x}_1, \bar{x}_2)$ subject to $\boxed{g(x_1, x_2) = 0}$. If $\nabla g(\bar{x}_1, \bar{x}_2) \neq 0$, then there exists some λ s.t.

$$\boxed{\nabla f(\bar{x}_1, \bar{x}_2) = \lambda \cdot \nabla g(\bar{x}_1, \bar{x}_2).}$$

This suggests that when we want to optimize $f(x_1, x_2)$ s.t. the constraint $g(x_1, x_2) = 0$, we have to solve the system

$$(*) \quad \boxed{\begin{aligned} g(x_1, x_2) &= 0 \\ \nabla f(x_1, x_2) &= \lambda \nabla g(x_1, x_2) \end{aligned}}$$

The number λ is called the <u>Lagrange multiplier</u>.

<u>Example</u>: Find all extrema of $\boxed{f(x_1, x_2) = e^{-x_1 x_2}}$ subject to the constraint $x_1^2 + 4x_2^2 = 1$.

- The constraint can be written as $g(x_1, x_2) = 0$, where $g(x_1, x_2) = x_1^2 + 4x_2^2 - 1$.
- We now solve the system $(*)$:

$$\nabla f(x_1, x_2) = \begin{pmatrix} -x_2 e^{-x_1 x_2} \\ -x_1 e^{-x_1 x_2} \end{pmatrix}$$

$$\nabla g(x_1, x_2) = \begin{pmatrix} 2x_1 \\ 8x_2 \end{pmatrix}$$

$$\stackrel{(*)}{\Longrightarrow} \quad \begin{aligned} x_1^2 + 4x_2^2 - 1 &= 0 \\ -x_2 e^{-x_1 x_2} &= \lambda \cdot 2x_1 \quad &| \cdot 4x_2 \\ -x_1 e^{-x_1 x_2} &= \lambda \cdot 8x_2 \quad &| \cdot x_1 \end{aligned}$$

$$\Longrightarrow \quad \begin{aligned} x_1^2 + 4x_2^2 - 1 &= 0 \\ -4x_2^2 e^{-x_1 x_2} &= 8\lambda x_1 x_2 \\ -x_1^2 e^{-x_1 x_2} &= 8\lambda x_1 x_2 \end{aligned}$$

$$\Longrightarrow \quad \begin{aligned} x_1^2 + 4x_2^2 - 1 &= 0 \\ x_1^2 \cancel{e^{-x_1 x_2}} &= 4x_2^2 \cancel{e^{-x_1 x_2}} \end{aligned}$$

$$\Longrightarrow \quad \begin{aligned} x_1^2 + 4x_2^2 &= 1 \\ x_1^2 &= 4x_2^2 \end{aligned} \quad \Longrightarrow \underbrace{4x_2^2 + 4x_2^2 = 1}$$

$$8x_2^2 = 1$$

$$\Longrightarrow x_2^2 = \frac{1}{8}$$

$$\Longrightarrow \boxed{x_2 = \pm \frac{1}{\sqrt{8}}}$$

$$\Longrightarrow x_1^2 = 4 \cdot x_2^2 = 4 \cdot \frac{1}{8} = \frac{1}{2}$$

$$\Longrightarrow \boxed{x_1 = \pm \frac{1}{\sqrt{2}}}$$

Therefore, the system (*) has 4 solutions:

$$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{8}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{8}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{8}}\right),$$

$$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{8}}\right).$$

We have to find the values of f on all these points:

$$f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{8}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{8}}\right)$$

$$= e^{-\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{8}}} = e^{-\frac{1}{\sqrt{2 \cdot 8}}}$$

$$= \underline{\underline{e^{-\frac{1}{4}}}}$$

$$\underline{f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{8}}\right)} = \underline{f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{8}}\right)}$$

$$= e^{\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{8}}} = \underline{\underline{e^{\frac{1}{4}}}}$$

The maximum value of f s.t. the constraint g is $e^{\frac{1}{4}}$.
Therefore, $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{8}}\right)$ and $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{8}}\right)$ are maxima.

The minimum value of f s.t. the constraint g is $e^{-\frac{1}{4}}$.
Therefore, $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{8}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{8}}\right)$ are minima.

<u>Example</u>: Among two positive numbers with product equal to 1, find those whose sum is minimum.

Of course, we could say that $x_1 \cdot x_2 = 1$ $\Rightarrow \boxed{x_2 = \frac{1}{x_1}}$, and the sum of the two numbers is

$$x_1 + x_2 = x_1 + \frac{1}{x_1}.$$

So, we need to find the minimum of $h(x_1) = x_1 + \frac{1}{x_1}$, $x_1 > 0$.

It is also possible to use Lagrange multipliers:

We want to minimize

$$f(x_1, x_2) = x_1 + x_2, \quad \underline{x_1, x_2 > 0}$$

subject to the constraint $x_1 \cdot x_2 = 1$.

We set $g(x_1 x_2) = x_1 \cdot x_2 - 1$.

Now we solve the system (*):

- $\nabla f(x_1, x_2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- $\nabla g(x_1, x_2) = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$

$$\overset{(*)}{\Longrightarrow} \left.\begin{aligned} x_1 x_2 - 1 &= 0 \\ \underline{1} &= \underline{\lambda \cdot x_2} \\ 1 &= \underline{\lambda x_1} \end{aligned}\right\} \begin{aligned} &\boxed{x_1 x_2 = 1} \\ &\boxed{\lambda x_1 = \lambda x_2} \end{aligned}$$

$$\Longrightarrow x_1 = x_2 \Longrightarrow x_1^2 = 1$$

$$\Longrightarrow \underline{\underline{x_1 = \pm 1}}, \quad \underline{\underline{x_2 = \pm 1}}.$$

Because $x_1, x_2 > 0$, we get the solution

$$(\bar{x}_1, \bar{x}_2) = (1, 1).$$

Thus, among all positive numbers x_1, x_2 with $x_1 \cdot x_2 = 1$, the sum $x_1 + x_2$ is minimum when $x_1 = x_2 = 1$.

System of Differential Equations

We have solved certain DE of the form

$$y'(t) = f(t, y)$$

(for example, $y' = y^2 - 2y + 2$).

Such a DE has a "general solution".

E.g., the DE $y' = y$ has the general solution

$$y(t) = c \cdot e^t \quad (c \in \mathbb{R} \text{ is a constant}).$$

When we choose some specific value of $c \in \mathbb{R}$, we get a particular solution, e.g., $y = 5 \cdot e^t$.

Also, if we solve the DE with an initial condition, we solve an "initial value problem", and this has a unique solution.

E.g., $y' = y$, $y(0) = 1$ has the solution

$y(t) = e^t$.

We now deal with systems of DE, i.e.

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_n'(t) \end{pmatrix} = \begin{pmatrix} f_1(t, y_1, \dots, y_n) \\ f_2(t, y_1, \dots, y_n) \\ \vdots \\ f_n(t, y_1, \dots, y_n) \end{pmatrix}$$

The unknown here is the vector

$$y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}.$$

<u>Here</u>:

$$\boxed{\begin{aligned} y_1'(t) &= a_{11} y_1(t) + \dots + a_{1n} y_n(t) \\ &\vdots \\ y_n'(t) &= a_{n1} y_1(t) + \dots + a_{nn} y_n(t) \end{aligned}}$$

$$\iff y'(t) = \frac{dy(t)}{dt} = A \cdot y(t)$$

with $A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}$.

Such a system is called a homogeneous linear first-order system of DEs with constant coefficients.

We will only deal with the case $n=2$, i.e.,

$$(**) \quad \boxed{\begin{aligned} y_1'(t) &= a_{11} y_1(t) + a_{12} y_2(t) \\ y_2'(t) &= a_{21} y_1(t) + a_{22} y_2(t) \end{aligned}}$$

$$\Leftrightarrow \quad y'(t) = A \cdot y(t), \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

The Direction Field

We consider the system $(**)$. On the $y_1 y_2$-plane, at each point (y_1, y_2), we draw the vector

$$\begin{pmatrix} a_{11} y_1 + a_{12} y_2 \\ a_{21} y_1 + a_{22} y_2 \end{pmatrix}.$$

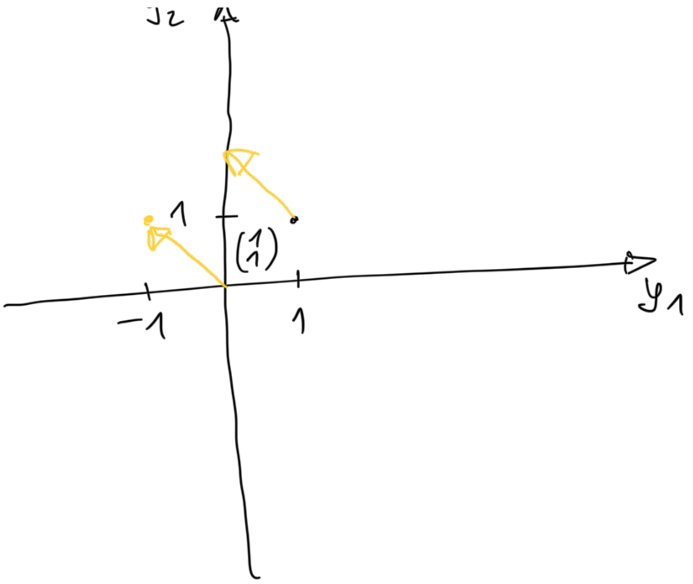
The collection of all these vectors is called the direction field of the system $(**)$.

Example: Consider the system

$$\begin{aligned} y_1' &= y_1 - 2y_2 \\ y_2' &= y_2 . \end{aligned}$$

Here, $A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$.

At $(y_1, y_2) = \binom{1}{1}$,



the vector is

$$\begin{pmatrix} y_1 + (-2)\, y_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 - 2 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}}$$

What does the direction field represent?

Suppose we want to solve the system

$$y' = Ay$$

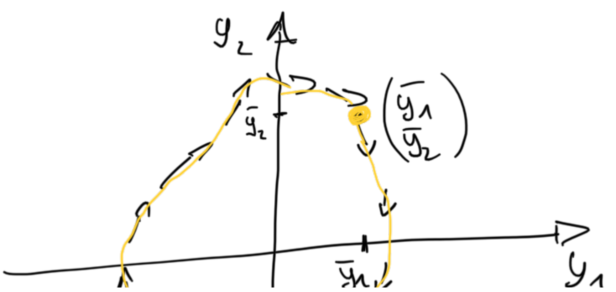
with initial condition $y(0) = \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}$.

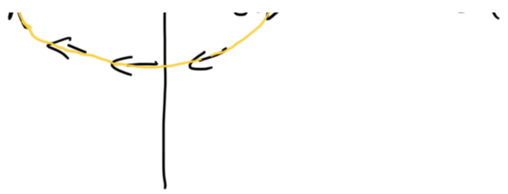
Then the unique solution will have the form

$$y_1 = y_1(t)$$
$$y_2 = y_2(t), \quad t \in \mathbb{R},$$

which represents a curve in the $y_1 y_2$-plan that goes through the point $\begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix}$.

At each point of the solution curve, the direction field is a tangent to the curve:





Solving the system $y'(t) = A \cdot y(t)$

- Assumption: A has two distinct eigenvalues λ_1, λ_2

- First, we find the two eigenvalues λ_1, λ_2 of A
- We find an eigenvector $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ corresponding to λ_1 and an eigenvector $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ corr. to λ_2.
- The general solution is:

$$\boxed{y(t) = c_1 e^{\lambda_1 t} u + c_2 e^{\lambda_2 t} v}$$

$$\Longleftrightarrow \boxed{\begin{aligned} y_1(t) &= c_1 e^{\lambda_1 t} u_1 + c_2 e^{\lambda_2 t} v_1 \\ y_2(t) &= c_1 e^{\lambda_1 t} u_2 + c_2 e^{\lambda_2 t} v_2, \end{aligned}}$$

$(t \in \mathbb{R})$

where $c_1, c_2 \in \mathbb{R}$ are constants.

Example: Solve $(**)$ with $A = \begin{pmatrix} 2 & -2 \\ 2 & -3 \end{pmatrix}$ and initial condition $y(0) = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$.

- Find the eigenvalues of A:

$\det(\lambda \cdot I - A) = \begin{vmatrix} \lambda - 2 & \boxed{2} \end{vmatrix}$

$$\det(\lambda I - A) = \begin{vmatrix} \cdots & \cdots \\ \boxed{-2} & \lambda+3 \end{vmatrix}$$

$$= (\lambda-2)(\lambda+3) + 4$$
$$= \lambda^2 + 3\lambda - 2\lambda - 6 + 4$$
$$= \lambda^2 + \lambda - 2$$
$$= (\lambda+2)(\lambda-1)$$

The eigenvalues of A are

$$\lambda_1 = -2, \quad \lambda_2 = 1.$$

- Find an eigenvector for $\lambda_1 = -2$:

$$A \cdot u = -2 \cdot u \implies \begin{pmatrix} 2 & -2 \\ 2 & -3 \end{pmatrix} \cdot u = \begin{pmatrix} -2u_1 \\ -2u_2 \end{pmatrix}$$

$$\implies \left.\begin{aligned} 4u_1 - 2u_2 &= 0 \\ 2u_1 - u_2 &= 0 \end{aligned}\right\} \quad \underline{2u_1 = u_2}.$$

One eigenvector is $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

- Find an eigenvector for $\lambda_2 = 1$:

$$Av = 1 \cdot v \implies \begin{pmatrix} 2 & -2 \\ 2 & -3 \end{pmatrix} \cdot v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\implies \left.\begin{aligned} v_1 - 2v_2 &= 0 \\ 2v_1 - 4v_2 &= 0 \end{aligned}\right\} \quad v_1 = 2v_2$$

One eigenvector is $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

• The general solution of the system is

$$y(t) = c_1 e^{\lambda_1 t} u + c_2 e^{\lambda_2 t} v$$

$$= c_1 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Longleftrightarrow \boxed{\begin{aligned} y_1(t) &= c_1 e^{-2t} + 2 c_2 e^{t} \\ y_2(t) &= 2 c_1 e^{-2t} + c_2 e^{t} \end{aligned}}$$

$(t \in \mathbb{R},\ c_1, c_2 \in \mathbb{R}$ constants$)$

• Specific solution:

$$y(\underset{\underset{t=0}{\uparrow}}{0}) = \begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} c_1 \cdot e^{0} + 2 c_2 \cdot e^{0} \\ 2 c_1 e^{0} + c_2 e^{0} \end{pmatrix}$$

$$= \begin{pmatrix} c_1 + 2 c_2 \\ 2 c_1 + c_2 \end{pmatrix}$$

So,
$$\left.\begin{aligned} -1 &= c_1 + 2 c_2 \\ 4 &= 2 c_1 + c_2 \end{aligned}\right\} \Rightarrow c_1 = -1 - 2 c_2$$

$$\Rightarrow 4 = 2 \cdot (-1 - 2 c_2) + c_2$$

$$= -2 - 4 c_2 + c_2$$

$$\Rightarrow 6 = -3 c_2 \Rightarrow c_2 = -2$$

$$\Rightarrow c_1 = -1 - 2 \cdot (-2) = 3$$

The unique solution is

$$y(t) = \begin{pmatrix} 3e^{-2t} - 4e^{t} \\ 6e^{-2t} - 2e^{t} \end{pmatrix}, \quad t \in \mathbb{R}$$