Global Minima and Maxima

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Some Definitions

Let $D \subseteq \mathbb{R}^2$ be a subset of the plane. The point $(\bar{x}_1, \bar{x}_2) \in D$ is called an **interior point** of D if there exists some $R > 0$ s.t. $B_R(\bar{x}_1, \bar{x}_2) \subseteq D$. The set $D \subseteq \mathbb{R}^2$ is called **open** if all its points are interior points.

Examples:

- $B_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$
- $S = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < 1,\ 0 < x_2 < 1\}$.

The set $D \subseteq \mathbb{R}^2$ is called **closed** if the complement $\mathbb{R}^2 \setminus D$ is open. The set $D \subseteq \mathbb{R}$ is called **bounded** if there exists some $R > 0$ s.t. D is contained in the disc with center $(0, 0)$ and radius R.

Existence of Global Minima and Maxima

Theorem

Let $D \subseteq \mathbb{R}^2$ be a closed and bounded subset of the plane and let $f : D \to \mathbb{R}$ be a continuous function. Then f has a global minimum and a global maximum.

How can find **global** minima and maxima of a function f?

- First we find local extrema of f in the interior of D. If f has a local extremum on $(\bar{x}_1, \bar{x}_2)$, then $\nabla f(\bar{x}_1, \bar{x}_2) = 0$.
- We find all values of f on the boundary of D.
- The global minima and maxima of f (if they exist) will necessarily be among the values that we found in the previous two steps.

Example

Find the local and global extrema of $f(x_1, x_2) = x_1^2 - 3x_2 + x_2^2$, where $-1 \le x_1 \le 1$, $0 \le x_2 \le 2$.

f is continuous and defined on a closed and bounded set, so f will have a global minimum and maximum on D. First we check for any extrema of f in the interior of D.

$$\begin{aligned} f_{x_1}(x_1, x_2) &= 2x_1 \\ f_{x_2}(x_1, x_2) &= -3 + 2x_2. \end{aligned}$$

Setting the gradient of f to zero (i.e., $\nabla f(x_1, x_2) = 0$) leads to

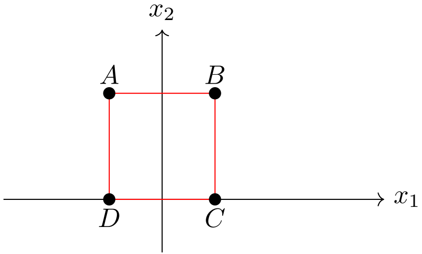
$$\begin{aligned} 2x_1 &= 0 \\ -3 + 2x_2 &= 0, \end{aligned}$$

which gives $x_1 = 0$ and $x_2 = \frac{3}{2}$. The function value at this point is

$$f\left(0, \frac{3}{2}\right) = 0 - 3 \cdot \frac{3}{2} + \frac{9}{4} = -\frac{9}{4}.$$

Example (cont.)

Now we find the values of f on the boundary of the domain.



- On DC: $x_2 = 0$ and $-1 \leq x_1 \leq 1$. f has the form $f(x_1, 0) = x_1^2$, $-1 \leq x_1 \leq 1$. At the points on $(1, 0)$ and $(-1, 0)$ on DC, the function value is 1.
- On CB: $x_1 = 1$ and $0 \leq x_2 \leq 2$. f has the form $f(1, x_2) = 1 - 3x_2 + x_2^2 = g(x_2)$.

$$g'(x_2) = 2x_2 - 3 = 0 \quad \Longrightarrow \quad x_2 = \frac{3}{2}.$$

So, $(1, \frac{3}{2})$ with $f(1, \frac{3}{2}) = -\frac{5}{4}$ is a candidate for a global extremum.

Example (cont.)

- On AB: $x_2 = 2$ and $-1 \leq x_1 \leq 1$. f has the form $f(x_1, 2) = x_1^2 - 2 = g(x_2)$. Then $g'(x_2) = 2x_1 = 0$ yields $x_2 = 0$ and thus $(0, 2)$ is a candidate for an extremum. Other candidates are the endpoints $(-1, 2)$ with $f(-1, 2) = -1$ and $(1, 2)$ with $f(1, 2) = -1$.
- On AD: $x_1 = -1$ and $0 \leq x_2 \leq 2$. f has the form $f(-1, x_2) = 1 - 3x_2 + x_2^2$ (the same as on BC). Thus we have the point $(-1, \frac{3}{2})$ with $f(-1, \frac{3}{2}) = -\frac{5}{4}$. Other candidates are the endpoints $(-1, 0)$ with $f(-1, 0) = 1$ and $(-1, 2)$ with $f(-1, 2) = -1$.

f has a global minimum at $(0, \frac{3}{2})$ with $f(0, \frac{3}{2}) = -\frac{9}{4}$.

f has a global maximum at $(-1, 0)$ and $(1, 0)$ with $f(1, 0) = f(1, 0) = 1$.

Example

Find the global extrema of $f(x_1, x_2) = x_1^2 + x_2^2 - 2x_1 + 4$ on the disc $D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 4\}$.

D is a closed and bounded set and f is continuous on D, so f will have a global minimum and maximum on D. First we check for any extrema of f in the interior of D.

$$\begin{aligned} f_{x_1}(x_1, x_2) &= 2x_1 - 2 \\ f_{x_2}(x_1, x_2) &= 2x_2. \end{aligned}$$

Setting the gradient of f to zero (i.e., $\nabla f(x_1, x_2) = 0$) leads to

$$\begin{aligned} 2x_1 - 2 &= 0 \\ 2x_2 &= 0, \end{aligned}$$

which gives $x_1 = 1$ and $x_2 = 0$. The function value at this point is

$$f(1, 0) = 1^2 - 2 + 4 = 3.$$

Example (cont.)

We now find the values of f on the boundary of D, which is the circle $x_1^2 + x_2^2 = 4$. We find a parametrisation of the points on the boundary of D: Points x_1, x_2 with $x_1^2 + x_2^2 = 4$ have coordinates of the form

$$\begin{aligned} x_1 &= 2\cos(\theta), \quad 0 \le \theta < 2\pi \\ x_2 &= 2\sin(\theta). \end{aligned}$$

Thus, on the boundary of D, we have

$$\begin{aligned} f(x_1, x_2) &= f(2\cos(\theta), 2\sin(\theta)) \\ &= \underbrace{4(\cos(\theta))^2 + 4(\sin(\theta))^2}_{(\cos(\theta))^2 + (\sin(\theta))^2 = 1} - 4\cos(\theta) + 4 \\ &= 8 - 4\cos(\theta) \\ &= 4(2 - \cos(\theta)) \\ &= g(\theta), 0 \le \theta < 2\pi. \end{aligned}$$

$g(\theta)$ is maximum when

$$\cos(\theta) = -1 \quad \Longleftrightarrow \quad \theta = \pi,$$

and then $g(\pi) = 12$.

Example (cont.)

$g(\theta)$ is minimum when

$$\cos(\theta) = 1 \quad \Longleftrightarrow \quad \theta = 0,$$

and then $g(0) = 4$.

f has a global min. at $(1,0)$ with $f(1,0) = 3$, and f has a global max. at $(-2,0)$ with $f(-2,0) = 12$.

Example

Find three nonnegative numbers whose sum is equal to 90 such that their product is maximum.

Let $x_1, x_2, x_3 \geq 0$ be the three nonnegative numbers. Because their sum is equal to 90, we write

$$x_1 + x_2 + x_3 = 90 \quad \Longrightarrow \quad x_3 = 90 - x_1 - x_2.$$

Their product is

$$x_1 x_2 x_3 = x_1 x_2 (90 - x_1 - x_2).$$

We need to maximize the function

$$f(x_1, x_2) = x_1 x_2 (90 - x_1 - x_2) \ \ (= 90 x_1 x_2 - x_1^2 x_2 - x_1 x_2^2)$$

with $x_1 \geq 0$, $x_2 \geq 0$ and

$$x_3 \geq 0 \quad \Longrightarrow \quad 90 - x_1 - x_2 \geq 0 \quad \Longrightarrow \quad x_1 + x_2 \leq 90.$$

Example (cont.)

We find potential global extrema in the interior of f:

$$\nabla f(x_1, x_2) = \begin{pmatrix} f_{x_1}(x_1, x_2) \\ f_{x_2}(x_1, x_2) \end{pmatrix} = \begin{pmatrix} 90x_2 - 2x_1x_2 - x_2^2 \\ 90x_1 - x_1^2 - 2x_1x_2 \end{pmatrix}.$$

Setting $\nabla f(x_1, x_2) = 0$ gives

$$\begin{aligned} 90x_2 - 2x_1x_2 - x_2^2 &= x_2(90 - 2x_1 - x_2) = 0 \\ 90x_1 - x_1^2 - 2x_1x_2 &= x_1(90 - x_1 - 2x_2) = 0. \end{aligned}$$

Since we are looking for potential extrema in the interior, we must have $x_1, x_2 > 0$. Then

$$\begin{aligned} 90 - 2x_1 - x_2 = 0 \quad &\Longrightarrow \quad 2x_1 + x_2 = 90 \\ 90 - x_1 - 2x_2 = 0 \quad &\Longrightarrow \quad x_1 + 2x_2 = 90 \end{aligned}$$

with solution $(x_1, x_2) = (30, 30)$. We have

$$f(30, 30) = 30 \cdot 30 \cdot (90 - 30 - 30) = 27000.$$

On any point (x_1, x_2) on the boundary of the domain of f, we have $f(x_1, x_2) = 0$. Hence, f has a global max. at $(30, 30)$ (and $x_3 = 30$).