Lecture 22

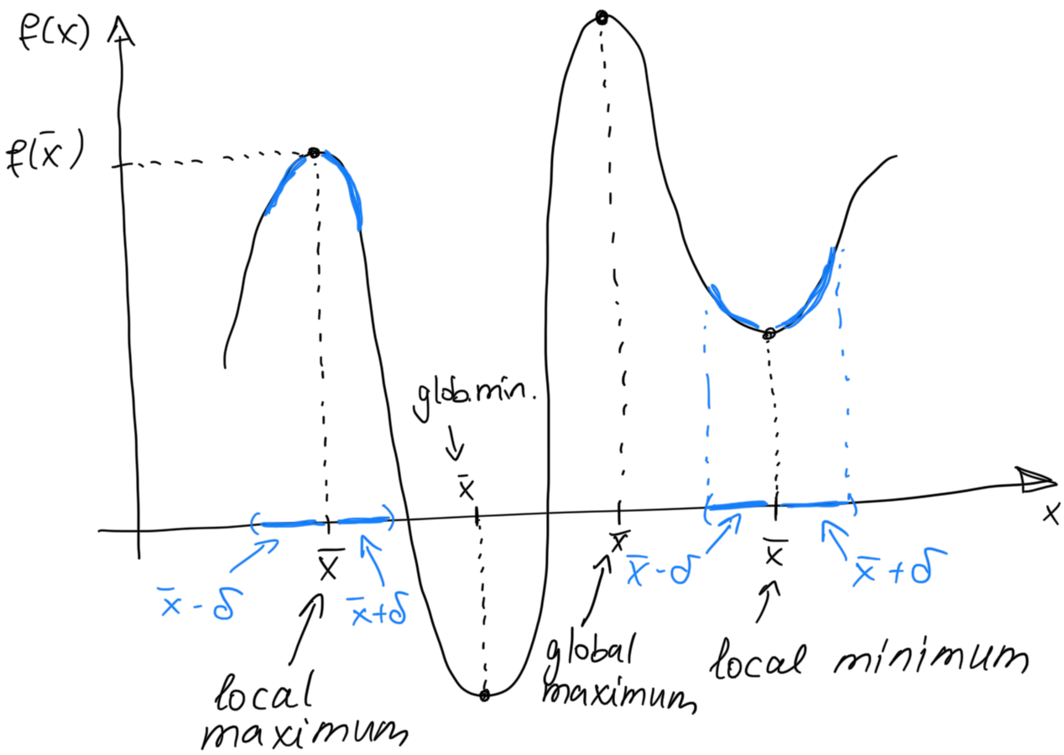
Minima and Maxima

We know how to find minima and maxima of functions of one variable.

Let $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}$ an interval.

We say that f has a <u>local maximum</u> at $\bar{x}$ if there is some $\delta > 0$ s.t.

$$\boxed{f(x) \leq f(\bar{x}) \text{ for all } x \in (\bar{x}-\delta, \bar{x}+\delta).}$$



We say that f has a <u>local minimum</u> at $\bar{x}$ if there is some $\delta > 0$ s.t.

$$\boxed{f(x) \geq f(\bar{x}) \text{ for all } x \in (\bar{x}-\delta, \bar{x}+\delta)}$$

$f(x) \geq f(\bar{x})$ for all $x \in (\bar{x} - \delta, \ldots$

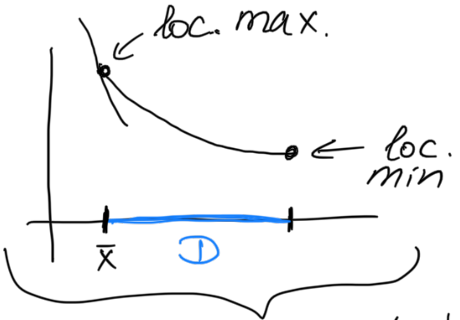
We say that f has a <u>global maximum</u> at $\bar{x}$ if

$$\boxed{f(x) \leq f(\bar{x}) \text{ for all } x \in D}$$

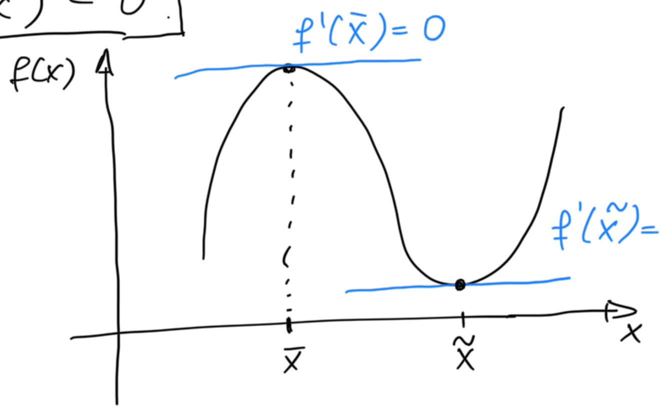
We say that f has a <u>global minimum</u> at $\bar{x}$ if

$$\boxed{f(x) \geq f(\bar{x}) \text{ for all } x \in D.}$$

<u>Fermat's Theorem</u>: If f has a local extremum at an interior point $\bar{x}$ and if $f'(\bar{x})$ exists, then $\boxed{f'(\bar{x}) = 0.}$

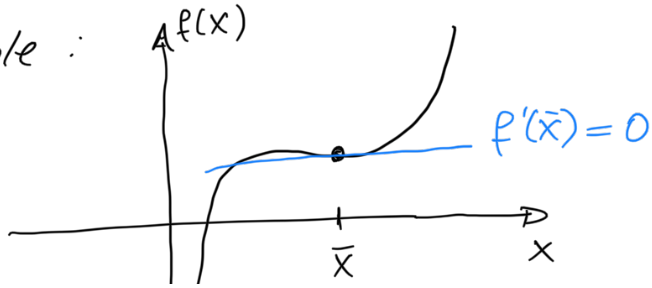


$\bar{x}$ is not an interior point and therefore, the above is not applicable.



<u>Remark</u>: The condition $f'(\bar{x}) = 0$ is necessary (and not sufficient).

For example:



$$\underbrace{\quad \| \quad}_{\bar{x} \text{ is not a local min. or loc. max., but } f'(\bar{x})=0.}$$

Points $\bar{x}$ in the domain of f where either
- f is not differentiable
- f is differentiable and $f'(\bar{x})=0$

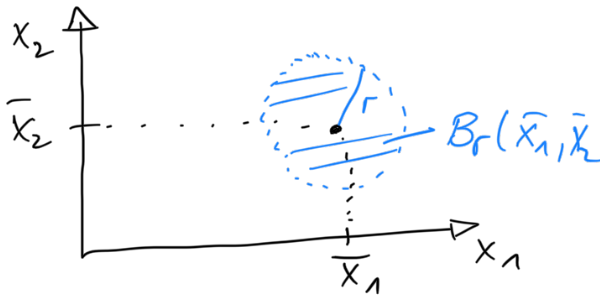
are called <u>critical points</u> of f.

Similar definitions hold for functions of more variables:

Let $f: D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$ and $(\bar{x}_1, \bar{x}_2) \in D$.
We say that f has a

- *local maximum* at $(\bar{x}_1,\bar{x}_2) \in D$ if there exists some $r>0$ s.t.

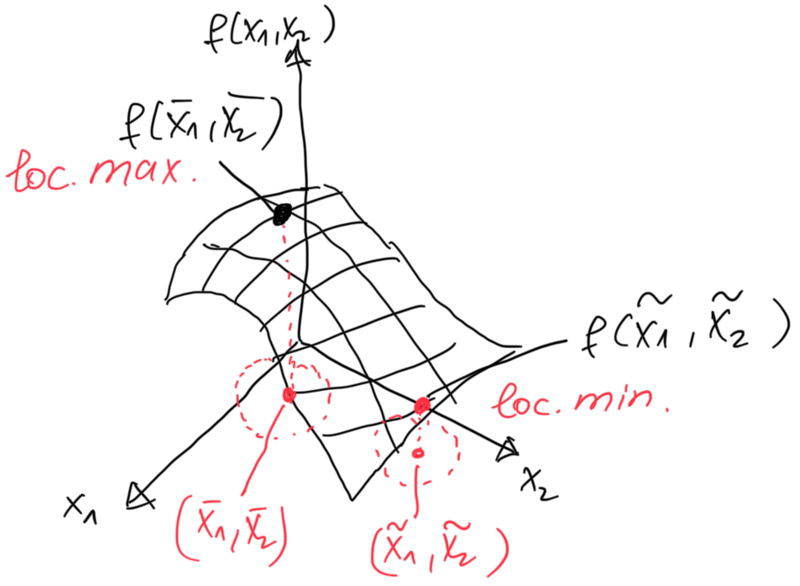
$$f(x_1,x_2) \leq f(\bar{x}_1,\bar{x}_2) \text{ for all } (x_1,x_2) \in \underbrace{B_r(\bar{x}_1,\bar{x}_2)}_{\text{= open disk with center } (\bar{x}_1,\bar{x}_2) \text{ and radius } r}$$



- *local minimum* at $(\bar{x}_1,\bar{x}_2) \in D$ if there exists some $r>0$ s.t.

$$f(x_1,x_2) \geq f(\bar{x}_1,\bar{x}_2) \text{ for all}$$

$(x_1, x_2) \in B_r(x_1, x_2)$.



We say that $(\bar{x}_1, \bar{x}_2)$ is a critical point of f if one of the following holds:

- f is not differentiable at $(\bar{x}_1, \bar{x}_2)$
- f is differentiable at $(\bar{x}_1, \bar{x}_2)$ and $\nabla f(\bar{x}_1, \bar{x}_2) = 0$.

<u>Theorem</u>: Let $f : D \to \mathbb{R}$, $D \subseteq \mathbb{R}^2$. If f has a local max. / local min. at the interior point $(\bar{x}_1, \bar{x}_2) \in D$ and f is differentiable at $(\bar{x}_1, \bar{x}_2)$, then

$$\boxed{\nabla f(\bar{x}_1, \bar{x}_2) = 0}$$

<u>Remark</u>: The theorem states that whenever f has a local max. (loc. min at $(\bar{x}_1, \bar{x}_2)$, then $\nabla f(\bar{x}_1, \bar{x}_2) = 0$.
If $\nabla f(\bar{x}_1, \bar{x}_2) = 0$, then this does not mean that $(\bar{x}_1, \bar{x}_2)$ will be necessarily

a local min. or loc. max.

<u>Example</u>: Let $f(x_1, x_2) = x_1^2 + x_2^2 + 1$.
This function has a local minimum at the point $(0,0)$.
(i) Prove that $\nabla f(0,0) = 0$.
(ii) Find the tangent plane of f at the point $(0,0)$.

(i) $$\left.\begin{aligned} f_{x_1}(x_1, x_2) &= 2x_1 \\ f_{x_2}(x_1, x_2) &= 2x_2 \end{aligned}\right\} \nabla f(x_1, x_2) = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$$

$$\Rightarrow \nabla f(0,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

(ii) The tangent plane at $(0,0)$ has the equation

$$y = f(0,0) + f_{x_1}(0,0) \cdot (x_1 - 0) + f_{x_2}(0,0) \cdot (x_2 - 0)$$

$$= 1 + 2 \cdot 0 \cdot x_1 + 2 \cdot 0 \cdot x_2$$

$$= \underline{\underline{1}}$$

$\Rightarrow$ The tangent plane at $(0,0)$ is $y = 1$ (the horizontal plane).

<u>Example</u>: Find all critical points of $f(x_1, x_2) = x_1^2 + x_2^2 + x_1 \cdot x_2$.

- f is differentiable in $\mathbb{R}^2$, so the critical points of f are those for which points of f ...

$$\boxed{\nabla f(x_1, x_2) = 0.}$$

- $f_{x_1}(x_1, x_2) = 2x_1 + x_2$
- $f_{x_2}(x_1, x_2) = 2x_2 + x_1$
- $\nabla f(x_1, x_2) = 0 \Rightarrow$ · $f_{x_1}(x_1, x_2) = 2x_1 + x_2 \overset{!}{=} 0$

$$\Rightarrow \boxed{x_2 = -2x_1}$$

· $f_{x_2}(x_1, x_2) = 2x_2 + x_1 \overset{!}{=} 0$

$$\Rightarrow 2 \cdot (-2x_1) + x_1$$
$$= -4x_1 + x_1$$
$$= -3x_1 \overset{!}{=} 0$$
$$\Rightarrow x_1 = 0$$
$$x_2 = 0.$$

$\Rightarrow$ The only critical point of f is the point $(0,0)$.
Does this function have a local extremum at $(0,0)$?
To answer this question, we consider the following theorem:

<u>Theorem</u>: Suppose that $f: D \to \mathbb{R}$ $(D \subseteq \mathbb{R}^2)$ has continuous partial derivatives of second order in some disc around $(\bar{x}_1, \bar{x}_2)$. Suppose also that $\nabla f(\bar{x}_1, \bar{x}_2) = 0$.

Let $\bar{D} := f_{x_1x_1}(\bar{x}_1, \bar{x}_2) \cdot f_{x_2x_2}(\bar{x}_1, \bar{x}_2)$

$(f_{x_1x_2}(\bar{x}_1, \bar{x}_2))^2$

$$-\left(f_{x_1x_2}(\bar{x}_1,\bar{x}_2)\right)^2.$$

Then:

- If $\bar{D} > 0$ and $\boxed{f_{x_1x_1}(\bar{x}_1,\bar{x}_2) > 0}$, then f has a <u>local minimum</u> at $(\bar{x}_1,\bar{x}_2)$.
- If $\bar{D} > 0$ and $\boxed{f_{x_1x_1}(\bar{x}_1,\bar{x}_2) < 0}$, then f has a <u>local maximum</u> at $(\bar{x}_1,\bar{x}_2)$.
- If $\bar{D} < 0$, then f does not have a local extremum at $(\bar{x}_1,\bar{x}_2)$. Then the point $(\bar{x}_1,\bar{x}_2)$ is called a <u>saddle point</u> of f.
- in any other case, the test is inconclusive

A way to remember this criterion: Compare with the second derivative test for a function $f: \mathbb{R} \to \mathbb{R}$: Suppose $\bar{x} \in \mathbb{R}$ is such that $f'(\bar{x}) = 0$.

- If $f''(\bar{x}) > 0$, then f has a local minimum at $\bar{x}$.
- If $f''(\bar{x}) < 0$, then f has a local maximum at $\bar{x}$.

<u>Example</u>: We found that $(0,0)$ is a critical point of the function

$$f(x_1,x_2) = x_1^2 + x_2^2 + x_1 \cdot x_2$$

Determine if $(0,0)$ is a local maximum local minimum.

- $f_{x_1}(x_1,x_2) = 2x_1 + x_2$

  $f_{x_1x_1}(x_1,x_2) = 2$

  $f_{x_1x_1}(0,0) = 2$

- $f_{x_2}(x_1,x_2) = 2x_2 + x_1$

  $f_{x_2x_2}(x_1,x_2) = 2$

  $f_{x_2x_2}(0,0) = 2$

- $f_{x_1x_2}(x_1,x_2) = 1 \quad (= f_{x_2x_1}(x_1,x_2))$

  $\Rightarrow f_{x_1x_2}(0,0) = 1.$

- $\bar{D} = f_{x_1x_1}(0,0) \cdot f_{x_2x_2}(0,0) - \left(f_{x_1x_2}(0,0)\right)^2$

  $= 2 \cdot 2 - 1^2 = 4 - 1 = 3 > 0$

and $f_{x_1x_1}(0,0) = 2 > 0$, so, the function f has a local minimum at $(0,0)$.

<u>Example</u>: Find all local extrema of

$$f(x_1,x_2) = 3x_1x_2 - x_1^3 - x_2^3,$$

$x_1, x_2 \in \mathbb{R}$.

- f has partial derivatives of all orders, so, if it has a local extremum at some point $(\bar{x}_1, \bar{x}_2)$, then

$$\boxed{\nabla f(\bar{x}_1, \bar{x}_2) = 0.}$$

- $f_{x_1}(x_1, x_2) = 3x_2 - 3x_1^2$

- $f_{x_2}(x_1, x_2) = 3x_1 - 3x_2^2$

$$\Rightarrow \nabla f(x_1, x_2) = \begin{pmatrix} 3x_2 - 3x_1^2 \\ 3x_1 - 3x_2^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} 3x_2 = 3x_1^2 \Rightarrow x_2 = x_1^2 \\ 3x_1 = 3x_2^2 \Rightarrow x_1 = x_2^2 \end{array}$$

We substitute the equation $x_1 = x_2^2$ into the first equation, and we get

$$x_2 = (x_2^2)^2 \Rightarrow x_2 = x_2^4$$

$$\Rightarrow x_2^4 - x_2 = 0 \Rightarrow x_2(x_2^3 - 1) = 0$$

$$\Rightarrow x_2 = 0 \quad \text{or} \quad x_2 = 1.$$

- When $x_2 = 0$, then $x_1 = 0$
- When $x_2 = 1$, then $x_1 = 1$.

$\Rightarrow$ The system $\nabla f(x_1, x_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has two solutions: $(0,0)$ and $(1,1)$.
These are the critical points of f.

- $f_{x_1 x_1}(x_1, x_2) = -6x_1$

- $f_{x_2x_2}(x_1,x_2) = -6x_2$
- $f_{x_1x_2}(x_1,x_2) = 3$

- Point $(0,0)$: $D = f_{x_1x_1}(0,0) \cdot f_{x_2x_2}(0,0) - (f_{x_1x_2}(0,0))^2$

$$= (-6 \cdot 0) \cdot (-6 \cdot 0) - 3^2$$
$$= -9 < 0$$

So, f has a saddle point at $(0,0)$.

- Point $(1,1)$:

$$D = f_{x_1x_1}(1,1) \cdot f_{x_2x_2}(1,1) - (f_{x_1x_2}(1,1))^2$$
$$= (-6 \cdot 1) \cdot (-6 \cdot 1) - 3^2$$
$$= 36 - 9 = 27 > 0,$$

and $f_{x_1x_1}(1,1) = -6 \cdot 1 = -6 < 0$,

so, f has a local maximum at $(1,1)$.

Example: Find the local extrema of
$$f(x_1,x_2) = 2x_1^2 - x_1x_2 + x_2^4.$$

- If f has a local extremum at $(\bar{x}_1,\bar{x}_2)$, then $\nabla f(\bar{x}_1,\hat{x}_2) = 0$.

- $f_{x_1}(x_1,x_2) = 4x_1 - x_2$
- $f_{x_2}(x_1,x_2) = -x_1 + 4x_2^3$
- $\nabla f(x_1,x_2) = 0 \Rightarrow \begin{array}{l} 4x_1 - x_2 = 0 \\ -x_1 + 4x_2^3 = 0 \end{array}$

$$\Rightarrow \begin{array}{l} x_2 = 4x_1 \\ -x_1 + 4\cdot(4x_1)^3 = -x_1 + 256x_1^3 \end{array}$$

$$= x_1(-1 + 256x_1^2) = 0$$

$$\Rightarrow x_1 = 0 \quad \text{or} \quad -1 + 256x_1^2 = 0$$

$$\Rightarrow 1 = 256x_1^2$$

$$\Rightarrow x_1^2 = \frac{1}{256}$$

$$\Rightarrow x_1 = \pm\sqrt{\frac{1}{256}}$$

$$= \pm\frac{1}{16}$$

- when $x_1 = 1$, $x_2 = 0$
- when $x_1 = \frac{1}{16}$, $x_2 = \frac{1}{4}$
- when $x_1 = -\frac{1}{16}$, $x_2 = -\frac{1}{4}$

$\Rightarrow$ The three critical points of f are:

$$(0,0),\ \left(\frac{1}{16}, \frac{1}{4}\right),\ \left(-\frac{1}{16}, -\frac{1}{4}\right).$$

- $f_{x_1x_1}(x_1,x_2) = 4$
- $f_{x_2x_2}(x_1,x_2) = 12x_2^2$

- $f_{x_1x_2}(x_1,x_2) = -1$
- <u>Point $(0,0)$</u>: $\bar{D} = f_{x_1x_1}(0,0)\cdot f_{x_2x_2}(0,0) - (f_{x_1x_2}(0,0$

$$= 4\cdot 12\cdot 0^2 - (-1)^2$$

$$= -1 < 0$$

$\Rightarrow$ f has a saddle point at $(0,0)$.

- <u>Point $\left(\frac{1}{16}, \frac{1}{4}\right)$</u>:

$$\bar{D} = f_{x_1x_1}\left(\tfrac{1}{16}, \tfrac{1}{4}\right)\cdot f_{x_2x_2}\left(\tfrac{1}{16}, \tfrac{1}{4}\right) - \left(f_{x_1x_2}\left(\tfrac{1}{16}, \tfrac{1}{4}\right)\right)^2$$

$$= \cancel{4}\cdot 12\cdot \frac{1}{4^{\cancel{2}}} - (-1)^2$$

$$= \frac{12}{4} - 1 = 3 - 1 = 2 > 0,$$

$f_{x_1x_1}\left(\frac{1}{16}, \frac{1}{4}\right) = 4 > 0$, so, f has a local minimum at $\left(\frac{1}{16}, \frac{1}{4}\right)$.

- <u>Point $\left(-\frac{1}{16}, -\frac{1}{4}\right)$</u>:

$$\bar{D} = f_{x_1x_1}\left(-\tfrac{1}{16}, -\tfrac{1}{4}\right)\cdot f_{x_2x_2}\left(-\tfrac{1}{16}, -\tfrac{1}{4}\right) - \left(f_{x_1x_2}\left(-\tfrac{1}{16}, -\tfrac{1}{4}\right)\right)^2$$

$$= 4\cdot 12\cdot\left(-\frac{1}{4}\right)^2 - (-1)^2$$

$= 3 - 1 = 2 > 0,$

$f_{x_1 x_1}\left(-\frac{1}{16}, -\frac{1}{4}\right) = 4 > 0$, so, f has a local minimum at $\left(-\frac{1}{16}, -\frac{1}{4}\right)$.