Gradient of a Function of Several Variables & Directional Derivative

Elisabeth Köbis, elisabeth.kobis@ntnu.no

Gradient of a Function

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function which has partial derivatives of first order. We define the **gradient** of f at (x_1, x_2) to be the vector

$$\nabla f(x_1, x_2) = \begin{pmatrix} f_{x_1}(x_1, x_2) \\ f_{x_2}(x_1, x_2) \end{pmatrix}.$$

Example

Let $f(x_1, x_2) = x_1^2 + x_1 x_2$. Then $f_{x_1} = 2x_1 + x_2$, $f_{x_2} = x_1$. Hence,

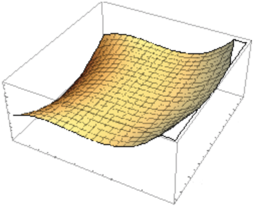
$$\nabla f(x_1, x_2) = \begin{pmatrix} 2x_1 + x_2 \\ x_1 \end{pmatrix}.$$

For example,

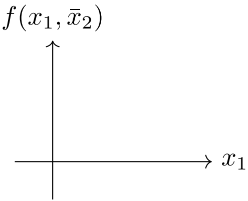
$$\nabla f(1, -1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Directional Derivatives

Recall the geometric interpretation of $\frac{\partial f(\bar{x}_1,\bar{x}_2)}{\partial x_1}$ and $\frac{\partial f(\bar{x}_1,\bar{x}_2)}{\partial x_2}$.



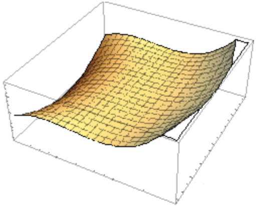
We keep x_2 constant and equal to $\bar{x}_2$. We intersect the graph of f with the plane $x_2 = \bar{x}_2$.



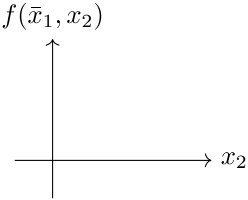
We get the graph of the function $g(x_1) = f(x_1, \bar{x}_2)$

The partial derivative $\frac{\partial f(\bar{x}_1,\bar{x}_2)}{\partial x_1}$ is the slope of the tangent at $(\bar{x}_1, f((\bar{x}_1, \bar{x}_2)))$.

Directional Derivatives



We keep x_1 constant and equal to $\bar{x}_1$. We intersect the graph of f with the plane $x_1 = \bar{x}_1$.

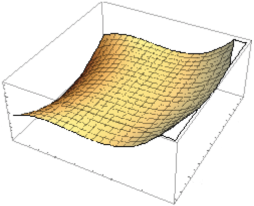


We get the graph of the function $g(x_2) = f(\bar{x}_1, x_2)$

The partial derivative $\frac{\partial f(\bar{x}_1, \bar{x}_2)}{\partial x_2}$ is the slope of the tangent at $(\bar{x}_2, f((\bar{x}_1, \bar{x}_2)))$.

Directional Derivatives

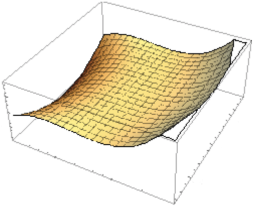
What if we want to find the rate of change of $f(x_1, x_2)$ as (x_1, x_2) changes along another direction?



Directional Derivative

Let $u = (u_1, u_2)$ be a unit vector (i.e., $|u| = \sqrt{u_1^2 + u_2^2 = 1}$). The **directional derivative** of $f(x_1, x_2)$ at the point $(\bar{x}_1, \bar{x}_2)$ in the direction of the unit vector $u = (u_1, u_2)$ is

$$D_u f(\bar{x}_1, \bar{x}_2) = \nabla f(\bar{x}_1, \bar{x}_2) \cdot u = u_1 \cdot \frac{\partial f(\bar{x}_1, \bar{x}_2)}{\partial x_1} + u_2 \cdot \frac{\partial f(\bar{x}_1, \bar{x}_2)}{\partial x_2}.$$



Directional Derivative

Example

Compute the directional derivative of

$$f(x_1, x_2) = \sqrt{x_1^2 + 2x_2^2}$$

at the point $(-1, 2)$ in the direction of $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

$f_{x_1}(x_1, x_2) = \frac{x_1}{\sqrt{x_1^2+2x_2^2}}$, $f_{x_2}(x_1, x_2) = \frac{2x_2}{\sqrt{x_1^2+2x_2^2}}$. This means

$$\nabla f(-1, 2) = \begin{pmatrix} f_{x_1}(-1, 2) \\ f_{x_2}(-1, 2) \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{4}{3} \end{pmatrix}.$$

The direction vector is the vector that we get if we normalize $v = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$:

$$|v| = \sqrt{(-1)^2 + 3^2} = \sqrt{1 + 9} = \sqrt{10}.$$

So, the direction vector is $u = \begin{pmatrix} \frac{-1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix}$.

Directional Derivative

The directional derivative of f at the point $(-1, 2)$ in the direction of u is

$$\begin{aligned} D_u f(-1,2) &= \nabla f(-1,2) \cdot u \\ &= \frac{1}{3\sqrt{10}} + \frac{12}{3\sqrt{10}} \\ &= \frac{13}{3\sqrt{10}}. \end{aligned}$$

Directional Derivative

Remark

Suppose we have the direction vector

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The directional derivative of f at $(\bar{x}_1, \bar{x}_2)$ w.r.t. u is

$$\begin{aligned} D_u f(\bar{x}_1, \bar{x}_2) &= \nabla f(\bar{x}_1, \bar{x}_2) \cdot u \\ &= \frac{\partial f(\bar{x}_1, \bar{x}_2)}{\partial x_1} \cdot 1 + \frac{\partial f(\bar{x}_1, \bar{x}_2)}{\partial x_2} \cdot 0 \\ &= \frac{\partial f(\bar{x}_1, \bar{x}_2)}{\partial x_1}, \text{ the partial derivative w.r.t. } x_1. \end{aligned}$$

Similarly, one can check that if $u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then the directional derivative of f at $(\bar{x}_1, \bar{x}_2)$ w.r.t. u is $\frac{\partial f(\bar{x}_1, \bar{x}_2)}{\partial x_2}$.

Directional Derivative

Example

Compute the directional derivative of $f(x_1, x_2) = x_1^2 x_2 - 2x_2^2$ at the point $(-3, 2)$ in the direction of $(-1, 1)$.

We have

$$f_{x_1} = 2x_1 x_2, \quad f_{x_2} = x_1^2 - 4x_2,$$

$$\nabla f(-3,2) = \begin{pmatrix} f_{x_1}(-3,2) \\ f_{x_2}(-3,2) \end{pmatrix} = \begin{pmatrix} -12 \\ 1 \end{pmatrix},$$

$$v = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \Longrightarrow \quad |v| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}.$$

This means that the unit vector at the direction of v is

$$u = \frac{v}{|v|} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}.$$

Hence, the directional derivative is

$$D_u f(-3,2) = \nabla f(-3,2) \cdot u = (-12) \cdot \left(-\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2} = \frac{13\sqrt{2}}{2}.$$

Directional Derivative

Suppose that $(\bar{x}_1, \bar{x}_2)$ is a point in the domain of f and u is a unit vector. The directional derivative at $(\bar{x}_1, \bar{x}_2)$ in the direction of u is

$$\begin{aligned} D_u(f(\bar{x}_1, \bar{x}_2)) &= \nabla f(\bar{x}_1, \bar{x}_2) \cdot u \\ &= |\nabla f(\bar{x}_1, \bar{x}_2)| \cdot \underbrace{|u|}_{=1} \cdot \cos\theta \\ &= |\nabla f(\bar{x}_1, \bar{x}_2)| \cdot \cos\theta. \end{aligned}$$

This means that the directional derivative is maximum when $\cos\theta = 1$, and u has the same direction as the gradient $\nabla f(\bar{x}_1, \bar{x}_2)$.

> The gradient $\nabla f(\bar{x}_1, \bar{x}_2)$ points towards the direction at which $f(x_1, x_2)$ increases most rapidly.

Directional Derivative

Recall that the level sets (or level curves) of a function $f(x_1, x_2)$ are the sets of the form

$$L_c = \{(x_1, x_2) \in \mathbb{R}^2 : f(x_1, x_2) = c\}.$$

If $(\bar{x}_1, \bar{x}_2)$ is a point on the level curve $L_c = \{(x_1, x_2) \in D : f(x_1, x_2) = c\}$, then the gradient of f, $\nabla f(\bar{x}_1, \bar{x}_2)$ at the point $(\bar{x}_1, \bar{x}_2)$ is perpendicular to the level curve at the point $(\bar{x}_1, \bar{x}_2)$.

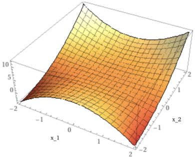
Directional Derivative

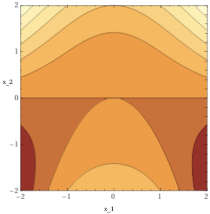
Example

Let $f(x_1, x_2) = x_1^2 x_2 + x_2^2$. What is the direction at the point $(1, 1)$ at which the function f increases most rapidly?

$$f_{x_1}(x_1, x_2) = 2x_1x_2, \quad f_{x_2}(x_1, x_2) = x_1^2 + 2x_2, \quad \nabla f(1, 1) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

The vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ shows the direction at which f increases most rapidly.





Directional Derivative

Example

Let $f(x_1, x_2) = x_1^2 - x_2^2$.

1. What is the level curve of f that goes though the point $(1, 2)$?
2. Find a vector perpendicular to this curve at the point $(1, 2)$.

1. We have $f(1, 2) = 1^2 - 2^2 = -3$, so the level curve of f that does through the point $(1, 2)$ is the curve

$$x_1^2 - x_2^2 = -3.$$

2. The requested vector is $\nabla f(1, 2)$:

$$\frac{\partial f}{\partial x_1} = 2x_1, \quad \frac{\partial f}{\partial x_2} = -2x_2,$$

and so $\nabla f(1, 2) = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$.