Chain Rule for Functions of Several Variables

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Chain Rule for 1-Variable Functions

For functions of one variable, we recall the **chain rule** as follows: If $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$h(x) = f(g(x)),$$

is well-defined, then the derivative of $h(x)$ is (in function notation)

$$h'(x) = f'(g(x)) \cdot g'(x).$$

Using the notation $\frac{df}{dx}$ for the derivative $f'(x)$ (where f is a function of g), the chain rule is written (in Leibniz notation) as

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}.$$

Example

If $h(x) = (2x-1)^3$, we have $f(x) = x^3$ and $g(x) = 2x - 1$. We get

$$h'(x) = f'(g(x)) \cdot g'(x) = 3 \cdot (2x-1)^2 \cdot 2,$$

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx} = 3 \cdot (2x-1)^2 \cdot 2.$$

Motivation for Introducing the Chain Rule for 2-Variable Functions: Application

Sometimes, we have to write functions of two or more variables in parametric form.

Suppose the total concentration C of CO_2 depends both on the temperature T and on the light intensity I. So, we have

$$C = C(T, I).$$

Now assume that T and I depend on the time t, so

$$T = T(t), \quad I = I(t).$$

Then C is a function of time as well:

$$C(t) = C(T(t), I(t)).$$

We know how to find $\frac{\partial C}{\partial T}$, $\frac{\partial C}{\partial I}$, and also $\frac{dT}{dt}$, $\frac{dI}{dt}$, but what about $\frac{dC(t)}{dt}$?

Chain Rule for 2-Variable Functions

Theorem

Suppose $f(x_1, x_2)$ is a differentiable function of two variables, and $x_1 = x_1(t)$, $x_2 = x_2(t)$ are differentiable as functions of t. Then

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \cdot \frac{dx_2}{dt}.$$

Example

Suppose $f(x_1, x_2) = x_1^2 x_2^3$ and $x_1 = x_1(t) = \sin(t)$, $x_2 = x_2(t) = \exp(-t)$. Find the derivative of $\frac{df}{dt}|_{t=\pi/2}$ using the chain rule.

$$\frac{df}{dt} = \frac{\partial f}{\partial x_1} \cdot \frac{dx_1}{dt} + \frac{\partial f}{\partial x_2} \cdot \frac{dx_2}{dt} = 2x_1 x_2^3 \cdot \cos(t) + 3x_1^2 x_2^2 \cdot (-\exp(-t)).$$

When $t = \pi/2$, $x_1(\pi/2) = \sin(\pi/2) = 1$, $x_2(\pi/2) = \exp(-\pi/2)$, and so

$$\frac{df}{dt}|_{t=\frac{\pi}{2}} = 2\cdot 1\cdot \exp(-3\pi/2)\cdot \underbrace{\cos(\pi/2)}_{=0} - 3\cdot 1\cdot \exp(-3\pi/2) = -3\exp(-3\pi/2).$$

Chain Rule for 2-Variable Functions

For the same functions as before, we can compute the derivative of $\frac{df}{dt}|_{t=\pi/2}$ by finding $f(t)$ first and the differentiating:

$$f(t) = f(x_1(t), x_2(t)) = x_1(t)^2 x_2(t)^3 = (\sin(t))^2 \exp(-3t).$$

$$\Longrightarrow f'(t) = 2\sin(t)\cos(t)\exp(-3t) - 3(\sin(t))^2 \exp(-3t).$$

The derivative of f at $t = \pi/2$ is

$$\begin{aligned} f'\left(\frac{\pi}{2}\right) &= 2\sin\left(\frac{\pi}{2}\right)\underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0}\exp\left(\frac{-3\pi}{2}\right) - 3(\underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1})^2\exp\left(\frac{-3\pi}{2}\right) \\ &= -3\exp\left(\frac{-3\pi}{2}\right). \end{aligned}$$