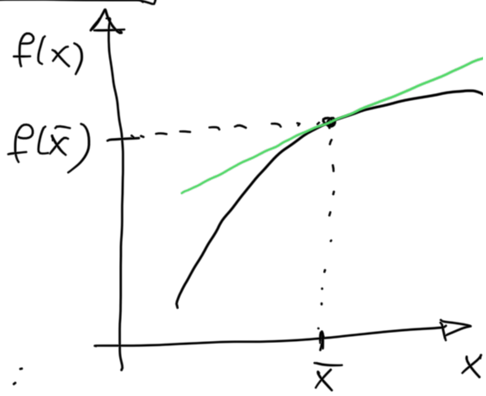
Lecture 20

Tangent Planes

Recall the notion of a tangent line of the graph of a function:

If $f: \mathbb{R} \to \mathbb{R}$ is differentiable at the point $\bar{x}$, the tangent line at the point $(\bar{x}, f(\bar{x}))$ of its graph has the following equation:

$$\boxed{y = f(\bar{x}) + f'(\bar{x})(x - \bar{x})}$$



- We know:
  - gradient of f at $\bar{x}$
  - line goes through the point $(\bar{x}, f(\bar{x}))$
- general formula for a line:

$$y = \underbrace{a}_{f'(\bar{x})} \cdot x + b \qquad \boxed{y = f'(\bar{x})x + f(\bar{x}) - f'}$$

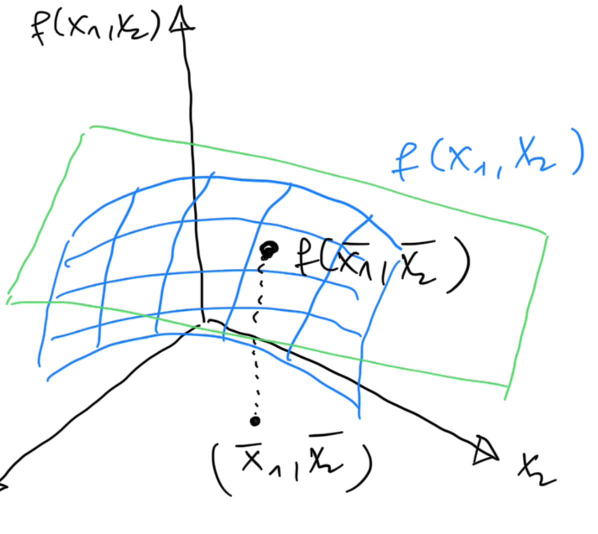
$$f(\bar{x}) = f'(\bar{x}) \cdot \bar{x} + b \implies b = f(\bar{x}) - f'(\bar{x})\cdot$$

For a function $f: D \to \mathbb{R}$ $(D \subseteq \mathbb{R}^2)$ of two variables, there exists the notion of the tangent plane.

Let $(\bar{x}_1, \bar{x}_2) \in D$ be a point in the

domain of f, and $\bar{y} = f(x_1, x_2)$.
Suppose that f has 1^{st}-order partial derivatives. Then the <u>tangent plane</u> of the graph of f at the point $(\bar{x}_1, \bar{x}_2, \bar{y})$ has the equation:

$$\boxed{y = f(\bar{x}_1, \bar{x}_2) + \frac{\partial f(\bar{x}_1, \bar{x}_2)}{\partial x_1}(x_1 - \bar{x}_1) + \frac{\partial f(\bar{x}_1, \bar{x}_2)}{\partial x_2}(x_2 - \bar{x}_2)}$$



<u>Example</u>: Find the tangent plane to the surface
$$f(x_1, x_2) = 4x_1^2 + x_2^2$$
at the point $(1, 2, 8)$.
$$\big((\bar{x}_1, \bar{x}_2) = (1, 2)\big)$$

- $\frac{\partial f(x_1, x_2)}{\ } = 8x_1$
- $\frac{\partial f(x_1, x_2)}{\partial x} = 2x_2$

∂x_1 ∂x_2

- $f(1,2) = 4 \cdot 1^2 + 2^2 = 8$
- $\dfrac{\partial f(1,2)}{\partial x_1} = 8 \cdot 1 = 8$ $\quad$ - $\dfrac{\partial f(1,2)}{\partial x_2} = 2 \cdot 2 = 4$

The tangent plane has the equation:

$$y = f(1,2) + \frac{\partial f(1,2)}{\partial x_1}(x_1 - 1) + \frac{\partial f(1,2)}{\partial x_2}(x_2 - 2)$$

$$= 8 + 8 \cdot (x_1 - 1) + 4(x_2 - 2)$$

$$= 8x_1 + 4x_2 - 8$$

Linear Approximation

When $f: D \to \mathbb{R}$ is a function of one variable ($D \subseteq \mathbb{R}$), the tangent line of f at $(\bar{x}, f(\bar{x}))$, given as $y = f(\bar{x}) + f'(\bar{x})(x - \bar{x})$, is also called the "best linear approximation" of f at the point $(\bar{x}, f(\bar{x}))$, i.e., the linear function y is the linear function that is closest to f near $\bar{x}$.

This linear approximation is also denoted by $L(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x})$.

Suppose that $f: D \to \mathbb{R}$ ($D \subseteq \mathbb{R}^2$) has ... derivatives f_{x_1}, f_{x_2}

continuous partial derivatives ... on some open disc centered at the point $(\bar{x}_1, \bar{x}_2) \in D$.

$\Rightarrow$ f has first order partial derivatives

The linear approximation of f at the point $(\bar{x}_1, \bar{x}_2)$ is the function

$$\boxed{L(x_1,x_2) = f(\bar{x}_1,\bar{x}_2) + f_{x_1}(\bar{x}_1,\bar{x}_2)(x_1-\bar{x}_1) + f_{x_2}(\bar{x}_1,\bar{x}_2)(x_2-\bar{x}_2)}$$

Example: Find the linear approximation of

$$f(x_1,x_2) = x_1^2 x_2 + 2x_1 e^{x_2}$$

at the point $(2,0)$.

- $f_{x_1} = 2x_1x_2 + 2e^{x_2}$ · $f_{x_1}(2,0) = 2\cdot 2\cdot 0 + 2\cdot \underbrace{e^0}_{=1} = 2$

- $f_{x_2} = x_1^2 + 2x_1 e^{x_2}$ · $f_{x_2}(2,0) = 2^2 + 2\cdot 2\cdot e^0 = 4+4 = 8$

- $f(2,0) = 4$

The linear approximation is

$$L(x_1,x_2) = f(2,0) + f_{x_1}(2,0)(x_1-2) + f_{x_2}(2,0)(x_2-0)$$

$$= 4 + 2\cdot(x_1-2) + 8x_2$$

$$= 2x_1 + 8x_2$$

Example: Find the linear approximation of $f(x_1, x_2) = \ln(x_1 - 2x_2^2)$ at the point $(3,1)$.

- $f_{x_1} = \dfrac{1}{x_1 - 2x_2^2}$ $\qquad$ - $f_{x_1}(3,1) = \dfrac{1}{3 - 2\cdot 1^2} = \underline{\underline{1}}$

- $f_{x_2} = \dfrac{-4x_2}{x_1 - 2x_2^2}$ $\qquad$ - $f_{x_2}(3,1) = \dfrac{-4}{3 - 2\cdot 1^2} = \underline{\underline{-4}}$

- $f(3,1) = \ln(3 - 2\cdot 1^2) = \ln 1 = \underline{\underline{0}}$

The linear approximation is

$$\underline{\underline{L(x_1, x_2)}} = f(3,1) + f_{x_1}(3,1)(x_1 - 3) + f_{x_2}(3,1)(x_2 - 1)$$

$$= 0 + 1\cdot(x_1 - 3) - 4(x_2 - 1)$$

$$= \underline{\underline{x_1 - 4x_2 + 1}}$$

Remark: The tangent plane is the graph of the linear approximation $L(x_1, x_2)$.

Vector-valued Functions

So far, we have only encountered functions $f: \mathbb{R}^n \to \mathbb{R}$, i.e., real-valued functions of multiple variables. That is, a point $(x_1, x_2, \dots, x_n)$ is mapped to some real number $f(x_1, x_2, \dots, x_n) \in \mathbb{R}$.

There also exist vector-valued functions $f: \mathbb{R}^n \to \mathbb{R}^m$ that map every point $(x_1, \dots, x_n) \in \mathbb{R}^n$ to some m-dimensional vector

$$f(x_1, \dots, x_n) = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ f_2(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{pmatrix}.$$

The functions $f_1, f_2, \dots, f_m : \mathbb{R}^n \to \mathbb{R}$ are called the <u>coordinate functions</u> of f.

For example, the function $f: \mathbb{R}^3 \to \mathbb{R}^3$,

$$f(x_1, x_2, x_3) = \begin{pmatrix} x_1^2 \\ x_2 x_3 \\ x_3 \cdot e^{-x_1} \end{pmatrix}$$

is a vector-valued function. The coordinate functio[ns] of f are

$$f_1(x_1, x_2, x_3) = x_1^2$$
$$f_2(x_1, x_2, x_3) = x_2 x_3$$
..

$$f_3(x_1, x_2, x_3) = x_3 \cdot e^{-x_1}.$$

No, we want to define the linear approximation of a vector-valued function.

For a real-valued function, its linear approximation at the point $(\bar{x}_1, \bar{x}_2)$ is

$$L(x_1, x_2) = f(\bar{x}_1, \bar{x}_2) + \underline{f_{x_1}(\bar{x}_1, \bar{x}_2)(x_1 - \bar{x}_1)}$$
$$+ \underline{f_{x_2}(\bar{x}_1, \bar{x}_2)(x_2 - \bar{x}_2)}$$

$$= f(\bar{x}_1, \bar{x}_2) + \underline{\underline{\left(f_{x_1}(\bar{x}_1, \bar{x}_2), f_{x_2}(\bar{x}_1, \bar{x}_2)\right)}} \circ$$
$$\circ \begin{pmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \end{pmatrix}.$$

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix}$.

We define the <u>Jacobi-matrix</u> of f at the point $(\bar{x}_1, \bar{x}_2)$ to be

$$(Df)(\bar{x}_1, \bar{x}_2) = \begin{pmatrix} (f_1)_{x_1}(\bar{x}_1, \bar{x}_2) & (f_1)_{x_2}(\bar{x}_1, \bar{x}_2) \\ (f_2)_{x_1}(\bar{x}_1, \bar{x}_2) & (f_2)_{x_2}(\bar{x}_1, \bar{x}_2) \end{pmatrix}$$

The linear approximation (= linearisation) of f at the point $(\bar{x}_1, \bar{x}_2)$ is the function $L : \mathbb{R}^2 \to \mathbb{R}^2$ given as

$$L(x_1,x_2) = f(\bar{x}_1,\bar{x}_2) + (Df)(\bar{x}_1,\bar{x}_2)\begin{pmatrix} x_1-\bar{x}_1 \\ x_2-\bar{x}_2 \end{pmatrix}$$

$$= \begin{pmatrix} f_1(\bar{x}_1,\bar{x}_2) \\ f_2(\bar{x}_1,\bar{x}_2) \end{pmatrix} + \begin{pmatrix} (f_1)_{x_1}(\bar{x}_1,\bar{x}_2) & (f_1)_{x_2}(\bar{x}_1,\bar{x}_2) \\ (f_2)_{x_1}(\bar{x}_1,\bar{x}_2) & (f_2)_{x_2}(\bar{x}_1,\bar{x}_2) \end{pmatrix} \cdot \begin{pmatrix} x_1-\bar{x}_1 \\ x_2-\bar{x}_2 \end{pmatrix}$$

<u>Example</u>: Assume that $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1,x_2) = \begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix}$ is a vector valued function with

$$f_1(x_1,x_2) = x_1^2 x_2 - x_2^3,$$
$$f_2(x_1,x_2) = 2x_1^3 x_2^2 + x_2.$$

(i) Find $(Df)(x_1,x_2)$.

(ii) Evaluate the Jacobi matrix of f at the point $(1,2)$.

(iii) Find the linear approximation of f at $(1,2)$.

(i) $(f_1)_{x_1} = 2x_1x_2$, $(f_1)_{x_2} = x_1^2 - 3x_2^2$,

$(f_2)_{x_1} = 6x_1^2x_2^2$, $(f_2)_{x_2} = 4x_1^3x_2 + 1$

The Jacobi matrix of f is

$$(Df)(x_1,x_2) = \begin{pmatrix} 2x_1x_2 & x_1 - 3x_2 \\ 6x_1^2x_2^2 & 4x_1^3x_2 + 1 \end{pmatrix}$$

(ii) $$(Df)(1,2) = \begin{pmatrix} 4 & -11 \\ 24 & 9 \end{pmatrix}$$

(iii) The linear approximation of f at $(1,2)$ is

$$L(x_1,x_2) = f(1,2) + (Df)(1,2)\begin{pmatrix} x_1-1 \\ x_2-2 \end{pmatrix}$$

$$= \begin{pmatrix} -6 \\ 10 \end{pmatrix} + \begin{pmatrix} 4 & -11 \\ 24 & 9 \end{pmatrix} \cdot \begin{pmatrix} x_1-1 \\ x_2-2 \end{pmatrix}$$

$$= \begin{pmatrix} -6 \\ 10 \end{pmatrix} + \begin{pmatrix} 4(x_1-1) - 11(x_2-2) \\ 24(x_1-1) + 9(x_2-2) \end{pmatrix}$$

$$= \begin{pmatrix} -6 + 4x_1 - 4 - 11x_2 + 22 \\ 10 + 24x_1 - 24 + 9x_2 - 18 \end{pmatrix}$$

$$= \begin{pmatrix} 4x_1 - 11x_2 + 12 \\ 24x_1 + 9x_2 - 32 \end{pmatrix}$$

The Jacobi matrix and linearization are also defined for vector-valued functions of higher dimension, i.e., $f : \mathbb{R}^n \to \mathbb{R}^m$.

$$\Rightarrow f(x_1,\dots,x_n) = \begin{pmatrix} f_1(x_1,\dots,x_n) \\ \vdots \\ f_m(x_1,\dots,x_n) \end{pmatrix}.$$

The Jacobi matrix of f at the point

The Jacobian matrix of f at
$(x_1, \dots, x_n)$ is defined as

$$(Df)(x_1,\dots,x_n) = \underbrace{\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}}_{\substack{m \text{ rows} \\ n \text{ columns}}},$$

where the partial derivatives $\frac{\partial f_i}{\partial x_j}$ are evaluated at the point $(x_1, \dots, x_n)$.

The linear approximation of f at $(\bar{x}_1, \dots, \bar{x}_n) \in \mathbb{R}^n$ is

$$L(x_1,\dots,x_n) = \underbrace{f(\bar{x}_1,\dots,\bar{x}_n)}_{m\times 1} + \underbrace{(Df)(\bar{x}_1,\dots,\bar{x}_n)}_{m\times n} \cdot \underbrace{\begin{pmatrix} x_1 - \bar{x}_1 \\ \vdots \\ x_n - \bar{x}_n \end{pmatrix}}_{n\times 1}$$

<u>Example</u>: $f: \mathbb{R}^3 \to \mathbb{R}^4$:

$$f(x_1,x_2,x_3) = \begin{pmatrix} f_1(x_1,x_2,x_3) \\ f_2(x_1,x_2,x_3) \\ f_3 \quad -\text{''}- \\ f_4 \quad -\text{''}- \end{pmatrix} = \begin{pmatrix} x_1 x_2 \\ x_2 x_3 \\ x_3 x_1 \\ x_1 x_2 x_3 \end{pmatrix}$$

$$(\bar{x}_1, \bar{x}_2, \bar{x}_3) = (1, 1, 1).$$

$$(Df)(x_1, x_2, x_3) = \begin{pmatrix} x_2 & x_1 & 0 \\ 0 & x_3 & x_2 \\ x_3 & 0 & x_1 \\ x_2 x_3 & x_1 x_3 & x_1 x_2 \end{pmatrix}$$

$$(Df)(1,1,1) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\underline{\underline{L(x_1, x_2, x_3)}} = f(1,1,1) + (Df)(1,1,1) \cdot \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \\ x_3 - 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 - 1 \\ x_2 - 1 \\ x_3 - 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + (x_1 - 1) + (x_2 - 1) \\ 1 + (x_2 - 1) + (x_3 - 1) \\ 1 + (x_1 - 1) + (x_3 - 1) \\ 1 + (x_1 - 1) + (x_2 - 1) + (x_3 - 1) \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} x_1 + x_2 - 1 \\ x_2 + x_3 - 1 \\ x_1 + x_3 - 1 \\ x_1 + x_2 + x_3 - 2 \end{pmatrix}}}$$