Lecture 2

(v) Find $\int_4^9 \frac{2}{x-3}\,dx$.

Set $u = x-3 \Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$

limits: $x=4 \Rightarrow u = 4-3 = 1$
$x = 9 \Rightarrow u = 9-3 = 6$

$\left(f(x) = \frac{2}{x} \right)$

$$\int_4^9 \frac{2}{x-3}\,dx = \int_1^6 \frac{2}{u}\,du = 2\cdot\left[\ln|u|\right]_1^6$$
$$= 2\cdot(\ln 6 - \underbrace{\ln 1}_{=0})$$
$$= 2\cdot \ln 6$$

(vi) Calculate $\int_1^e \frac{\ln|x|}{x}\,dx$

Substitution: $u = \ln|x| \rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow du = \frac{1}{x}\cdot dx$

$$\Rightarrow \int_1^e \frac{\ln|x|}{x}\,dx = \int_{u(1)=\ln(1)=0}^{u(e)=\ln e = 1} u\,du = \left[\frac{1}{2}u^2\right]_0^1$$
$$= \frac{1}{2}$$

2. Integration by Parts

- product rule in integration form
- Let $u = u(x)$ and $v = v(x)$ be differentiable functions.

Then
$$(u\cdot v)' = u'v + u\cdot v'$$
$$\Leftrightarrow \quad uv' = (u\cdot v)' - u'v.$$

Integrating on both sides w.r.t. x:

integrating -

$$\int u \cdot \underbrace{v'}_{\frac{dv}{dx}} dx = \underbrace{\int (uv)' dx}_{= uv + C} - \int u' v \, dx$$

$$= uv - \int \underbrace{u'}_{\frac{du}{dx}} v \, dx$$

$$\Rightarrow \boxed{\int u \, dv = uv - \int v \, du}$$

$$\Leftrightarrow \boxed{\int u(x) \cdot v'(x) \, dx = \boxed{u(x) \cdot v(x)} - \boxed{\int v(x) u'(x) \, dx}}$$

(= rule for integration by parts)

Examples:

(i) Find $\int x \cdot \sin x \, dx$.

Let $u = x \quad u' = 1$

$v = -\cos x \quad v' = \sin x$

$$\Rightarrow \int x \sin x \, dx = -x \cdot \cos x - \int (-\cos x) \cdot 1 \, dx$$

$$= -x \cos x + \int \cos x \, dx$$

$$= -x \cos x + \sin x + C$$

(ii) Find $\int x \cdot \ln|x| \, dx$.

Choose: $u = \ln|x| \quad u' = \frac{1}{x}$

$v = \frac{1}{2} x^2 \quad v' = x$

$$\Rightarrow \int \ln|x| \cdot x \, dx = \ln|x| \cdot \frac{1}{2} x^2 - \int \frac{1}{2} x^{\cancel{2}} \cdot \frac{1}{\cancel{x}} \, dx$$

$$= \ln|x| \cdot \frac{1}{2} x^2 - \frac{1}{2} \int x \, dx$$

$$= \ln|x| \cdot \frac{1}{2} x^2 - \frac{1}{2} \cdot \frac{1}{2} x^2 + C$$

... by Parts - Formula for Definite Integrals

Integration by ...

When we calculate definite integrals, the method of integration by parts looks as follows:

$$\boxed{\int_a^b u(x)\,v'(x)\,dx = \big[u(x)v(x)\big]_a^b - \int_a^b v(x)\,u'(x)\,dx}$$

Example: Compute: $\int_0^1 x \cdot e^{-x}\,dx$

choose $u = x \quad u' = 1$

$v = -e^{-x} \quad v' = e^{-x}$

$$\Rightarrow \int_0^1 x \cdot e^{-x}\,dx = \big[-xe^{-x}\big]_0^1 - \int_0^1 1 \cdot (-e^{-x})\,dx$$

$$= \underbrace{-e^{-1}}_{= -\frac{1}{e}} + \int_0^1 e^{-x}\,dx$$

$$= -\frac{1}{e} - \big[e^{-x}\big]_0^1$$

$$= -\frac{1}{e} - (\underbrace{e^{-1}}_{=\frac{1}{e}} - \underbrace{e^{0}}_{=1})$$

$$= -\frac{2}{e} + 1$$

"Trick": Multiplying with 1:

Suppose we want to find $\int u(x)\,dx$. Then we can use our formula for integration by parts:

$$\boxed{\int u(x)\underbrace{v'(x)}_{=1}\,dx = u(x)\underbrace{v(x)}_{=x} - \int u'(x)\underbrace{v(x)}_{=x}\,dx}$$

Examples: (i) Find $\int \ln|x|\,dx$.

Choose: $u = \ln|x| \qquad u' = \frac{1}{x}$

$v = x \qquad v' = 1$

$$\int \ln|x|\,dx = x\cdot\ln|x| - \int \frac{1}{\cancel{x}}\cdot\cancel{x}\,dx$$
$$= x\cdot\ln|x| - \int 1\,dx$$
$$= x\cdot\ln|x| - x + C$$

(ii) Find $\int \underbrace{\tan^{-1}x}_{=\arctan x}\,dx$

Set $u = \arctan x \qquad u' = \frac{1}{x^2+1}$

$v = x \qquad v' = 1$

$$\Rightarrow \int \arctan x\,dx = x\cdot\arctan x - \int \frac{x}{x^2+1}\,dx$$

· Substitution for solving $\int \frac{x}{x^2+1}\,dx$:

$$\bar{u} = x^2 \Rightarrow \frac{d\bar{u}}{dx} = 2x \Rightarrow dx = \frac{d\bar{u}}{2x}$$

$$\Rightarrow \int \frac{x}{x^2+1}\,dx = \int \frac{\cancel{x}}{\bar{u}+1}\cdot\frac{d\bar{u}}{2\cancel{x}}$$
$$= \frac{1}{2}\cdot\int \frac{1}{\bar{u}+1}\,du$$
$$= \frac{1}{2}\cdot\ln|\bar{u}+1| + C$$
$$= \frac{1}{2}\cdot\ln|x^2+1| + C$$

Hence, we obtain:

$$\int \arctan x\,dx = x\cdot\arctan x - \frac{1}{2}\ln|x^2+1| + C$$

Using Integration by Parts Repeatedly

In some cases, integration by parts has to be used repeatedly (more than once):

Examples: (i) Compute $\int_0^1 x^2 e^x \, dx$.

Set $u = x^2 \qquad u' = 2x$

$v = e^x \qquad v' = e^x$

$$\Rightarrow \int_0^1 x^2 e^x \, dx = \left[x^2 \cdot e^x\right]_0^1 - 2\int_0^1 e^x \cdot x \, dx$$

Using integration by parts once more:

Set $\bar{u} = x \qquad \bar{u}' = 1$

$\bar{v} = e^x \qquad \bar{v}' = e^x$

$$\Rightarrow 2\int_0^1 e^x x \, dx = 2\left(\left[x \cdot e^x\right]_0^1 - \int_0^1 e^x \cdot 1 \, dx\right)$$

$$= 2\left[x e^x\right]_0^1 - 2\left[e^x\right]_0^1$$

$$= \cancel{2e} - (\cancel{2e} - 2 \cdot \underbrace{e^0}_{=1})$$

$$= \underline{\underline{2}}$$

So: $\int_0^1 x^2 e^x \, dx = \left[x^2 e^x\right]_0^1 - 2$

$$= \underline{\underline{e - 2}}$$

(ii) Evaluate $\int e^x \cos x \, dx$.

Set $u = \cos x \qquad u' = -\sin x$

$v = e^x \qquad v' = e^x$

$$\Rightarrow \int e^x \cos x \, dx = \cos x \cdot e^x - \int e^x \cdot (-\sin x) \, dx$$

$$= \underline{\cos x \, e^x} + \int e^x \sin x \, dx$$

Use integration by parts once more:

Set $\bar{u} = \sin x$ $\qquad \bar{u}' = \cos x$

$\bar{v} = e^x$ $\qquad \bar{v}' = e^x$

$$\Rightarrow \int e^x \sin x \, dx = \underline{e^x \sin x - \int e^x \cos x \, dx}$$

Therefore :

$$\boxed{\int e^x \cos x \, dx} = \underline{\cos x \, e^x} + \underline{e^x \sin x - \boxed{\int e^x \cos x \, dx}}$$

$$\Rightarrow 2 \int e^x \cos x \, dx = (\cos x + \sin x) e^x + c_1 \quad \Big| \text{ divide by } 2$$

$$\Rightarrow \int e^x \cos x \, dx = \underline{\underline{\frac{1}{2} (\cos x + \sin x) e^x + c}}$$

$$\frac{c_1}{2} = c$$

<u>Practical advice</u> : In integrals of the form

- $\int P(x) \cdot \sin(ax) \, dx$
- $\int P(x) \cdot \cos(ax) \, dx$
- $\int P(x) \cdot e^{ax} \, dx$,

where $P(x)$ is a polynomial ($P(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$) and a is a constant, (a_i : constants)

the polynomial $P(x)$ should be considered as u and the expressions $\sin(ax)$, $\cos(ax)$ and e^{ax} as v'.

If an integral contains the functions $\ln|x|$, $\arctan x$, or $\arcsin$, the function is usually treated as u.

- <u>Rational Functions</u> : A rational function is the quotient of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)},$$

where P and Q are polynomials. To integrate rational functions, we will use the method of partial fractions to write $f(x)$ as the sum of a polynomial and a simple rational function.

Example: Find $\int_1^3 \frac{6x^2}{(1-4x^3)^3}\,dx$

Substitution: $\boxed{u = 1 - 4x^3}$ $\qquad \frac{du}{dx} = -12x^2$

$\Rightarrow dx = \boxed{-\frac{1}{12x^2}\,du}$

$$\int_1^3 \frac{6x^2}{(1-4x^3)^3}\,dx = \frac{1}{2}\int_1^3 \frac{12x^2}{(1-4x^3)^3}\,\boxed{dx}$$

$$= \frac{1}{2}\int_{u(1)}^{u(3)} \frac{\cancel{12x^2}}{\boxed{u^3}}\left(-\frac{1}{\cancel{12x^2}}\right)du$$

$$= -\frac{1}{2}\int_{-3}^{-107} \frac{1}{u^3}\,du$$

$$= \frac{1}{2}\int_{-107}^{-3} \frac{1}{u^3}\,du$$

$$= \frac{1}{2}\left[-\frac{1}{2}u^{-2}\right]_{-107}^{-3}$$

$$= -\frac{1}{4}\left(\frac{1}{3^2} - \frac{1}{107^2}\right)$$