Partial Derivatives

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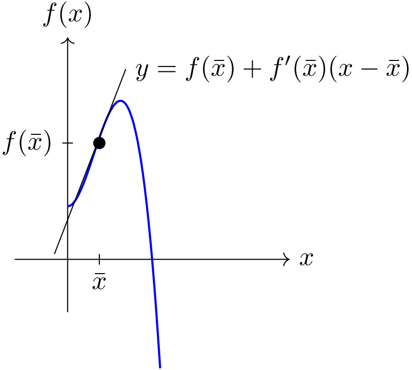
Derivatives of 1-Variable Functions

For functions f of one variable, we say that f is **differentiable** at the point $\bar{x}$ with derivative equal to $f'(\bar{x})$ if

$$\lim_{x \to \bar{x}} \frac{f(x) - f(\bar{x})}{x - \bar{x}} = \lim_{h \to 0} \frac{f(\bar{x} + h) - f(\bar{x})}{h} = f'(\bar{x}).$$

In that case, the number $f'(\bar{x})$ is the slope of the tangent at the graph of f at the point $(\bar{x}, f(\bar{x}))$.



Derivatives of 1-Variable Functions

The function which assigns to each value of x the derivative $f'(x)$ is called the **first derivative** of f.

Example

When $f(x) = x^2 + 1 + \exp(x)$, then its first derivative is the function

$$f'(x) = 2x + \exp(x).$$

Explanation: We get

$$\begin{aligned}
(x^2)'\Big|_{x=\bar{x}} &= \lim_{h\to 0} \frac{(\bar{x}+h)^2 - \bar{x}^2}{h} \\
&= \lim_{h\to 0} \frac{\bar{x}^2 + 2\bar{x}h + h^2 - \bar{x}^2}{h} \\
&= \lim_{h\to 0} \frac{2\bar{x}h + h^2}{h} = \lim_{h\to 0} \left(\frac{2\bar{x}h}{h} + \frac{h^2}{h}\right) \\
&= \lim_{h\to 0} (2\bar{x} + h) = 2\bar{x},
\end{aligned}$$

Derivatives of 1-Variable Functions

$$(1)' = \lim_{h \to 0} \frac{1 - 1}{h} = 0$$

and

$$\begin{aligned}
(\exp(\bar{x}))' &= \lim_{h \to 0} \frac{\exp(\bar{x} + h) - \exp(\bar{x})}{h} \\
&= \lim_{h \to 0} \frac{\exp(\bar{x}) \exp(h) - \exp(\bar{x})}{h} \\
&= \lim_{h \to 0} \frac{\exp(\bar{x})(\exp(h) - 1)}{h} \\
&= \exp(\bar{x}) \underbrace{\lim_{h \to 0} \frac{\exp(h) - 1}{h}}_{=1} \\
&= \exp(\bar{x}).
\end{aligned}$$

Derivatives of 2-Variable Functions

Assume we have a function $f(x_1, x_2)$ of two variables. If we treat one of the two variables as a constant, we can differentiate with respect to the other variable (as if we had a function of one variable).

Example

Let $f(x_1, x_2) = x_1 x_2 + x_1^3 x_2^2$. If we treat x_2 as a constant, we have a function of x_1, which is

$$g(x_1) = f(x_1, x_2) = x_1 x_2 + x_1^3 x_2^2.$$

Differentiating g, we get

$$g'(x_1) = x_2 + 3x_1^2 x_2^2.$$

Partial Derivatives

We define the **partial derivative** of $f(x_1, x_2)$ **with respect to** x_1 at the point $(\bar{x}_1, \bar{x}_2)$ to be the number

$$\frac{\partial f(\bar{x}_1, \bar{x}_2)}{\partial x_1} = \lim_{h \to 0} \frac{f(\bar{x}_1 + h, \bar{x}_2) - f(\bar{x}_1, \bar{x}_2)}{h},$$

and the **partial derivative** of $f(x_1, x_2)$ **with respect to** x_2 at the point $(\bar{x}_1, \bar{x}_2)$ is defined as

$$\frac{\partial f(\bar{x}_1, \bar{x}_2)}{\partial x_2} = \lim_{h \to 0} \frac{f(\bar{x}_1, \bar{x}_2 + h) - f(\bar{x}_1, \bar{x}_2)}{h},$$

assuming that these limits exist and are real numbers.

Partial Derivatives

The functions that map each point (x_1, x_2) to the partial derivative of f w.r.t. x_1 (x_2, respectively) are denoted by

$$\frac{\partial f(x_1, x_2)}{\partial x_1} \quad \text{and} \quad \frac{\partial f(x_1, x_2)}{\partial x_2}$$

and are also called the **partial derivatives** of f w.r.t. x_1 (x_2, respectively).

Sometimes, we write $f_{x_1}(x_1, x_2)$ instead of $\frac{\partial f(x_1,x_2)}{\partial x_1}$ and $f_{x_2}(x_1, x_2)$ instead of $\frac{\partial f(x_1,x_2)}{\partial x_2}$.

Partial Derivatives

Example

Find the partial derivatives $\frac{\partial f(x_1,x_2)}{\partial x_1}$, $\frac{\partial f(x_1,x_2)}{\partial x_2}$ when $f(x_1, x_2) = x_2 \exp(x_1 x_2)$.

$$\begin{aligned}
\frac{\partial f(x_1, x_2)}{\partial x_1} &= x_2^2 \exp(x_1 x_2) \\
\frac{\partial f(x_1, x_2)}{\partial x_2} &= \frac{\partial x_2}{\partial x_2} \cdot \exp(x_1 x_2) + x_2 \frac{\partial \exp(x_1 x_2)}{\partial x_2} \\
&= \exp(x_1 x_2) + x_1 x_2 \exp(x_1 x_2) = (1 + x_1 x_2) \exp(x_1 x_2).
\end{aligned}$$

Partial Derivatives

Example

Find the partial derivatives $\frac{\partial f(x_1,x_2)}{\partial x_1}$, $\frac{\partial f(x_1,x_2)}{\partial x_2}$ when $f(x_1, x_2) = \frac{\sin(x_1 x_2)}{x_1^2+\cos(x_2)}$.

$$\begin{aligned}\frac{\partial f(x_1, x_2)}{\partial x_1} &= \frac{u_{x_1} v - v_{x_1} u}{v^2} \\ &= \frac{x_2 \cos(x_1 x_2)(x_1^2 + \cos(x_2)) - 2x_1 \sin(x_1 x_2)}{(x_1^2 + \cos(x_2))^2}\end{aligned}$$

$$u = \sin(x_1 x_2) \quad u_{x_1} = x_2 \cos(x_1 x_2)$$

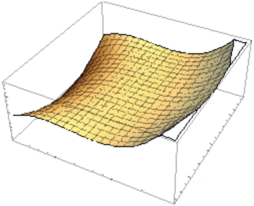
$$v = x_1^2 + \cos(x_2) \quad v_{x_1} = 2x_1$$

Partial Derivatives

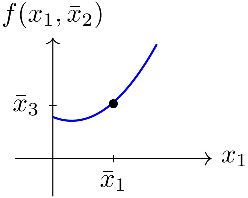
$$\frac{\partial f(x_1, x_2)}{\partial x_2} = \frac{u_{x_2} v - v_{x_2} u}{v^2}$$
$$= \frac{x_1 \cos(x_1 x_2)(x_1^2 + \cos(x_2)) - (-\sin(x_2)) \sin(x_1 x_2)}{(x_1^2 + \cos(x_2))^2}$$
$$= \frac{x_1 \cos(x_1 x_2)(x_1^2 + \cos(x_2)) + \sin(x_2) \sin(x_1 x_2)}{(x_1^2 + \cos(x_2))^2}$$

$$u = \sin(x_1 x_2) \quad u_{x_2} = x_1 \cos(x_1 x_2)$$
$$v = x_1^2 + \cos(x_2) \quad v_{x_2} = -\sin(x_2)$$

Partial Derivatives – Geometric Interpretations



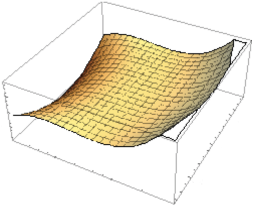
We keep x_2 constant and equal to $\bar{x}_2$. We intersect the graph of f with the plane $x_2 = \bar{x}_2$.



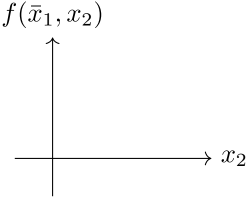
We get the graph of the function $g(x_1) = f(x_1, \bar{x}_2)$

The partial derivative $\frac{\partial f(\bar{x}_1, \bar{x}_2)}{\partial x_1}$ is the slope of the tangent at $(\bar{x}_1, f((\bar{x}_1, \bar{x}_2)))$.

Partial Derivatives – Geometric Interpretations



We keep x_1 constant and equal to $\bar{x}_1$. We intersect the graph of f with the plane $x_1 = \bar{x}_1$.



We get the graph of the function $g(x_2) = f(\bar{x}_1, x_2)$

The partial derivative $\frac{\partial f(\bar{x}_1,\bar{x}_2)}{\partial x_2}$ is the slope of the tangent at $(\bar{x}_2, f((\bar{x}_1, \bar{x}_2)))$.

Partial Derivatives for Functions of More Than 2 Variables

Partial Derivatives are also defined similarly for functions of more than two variables.

Example

$f(x_1, x_2, x_3) = \exp(x_2 x_3)(x_1^2 + x_3^3)$. Then

$$\frac{\partial f(x_1, x_2, x_3)}{\partial x_1} = 2x_1 \exp(x_2 x_3)$$
$$\frac{\partial f(x_1, x_2, x_3)}{\partial x_2} = x_3 \exp(x_2 x_3)(x_1^2 + x_3^3)$$
$$\frac{\partial f(x_1, x_2, x_3)}{\partial x_3} = x_2 \exp(x_2 x_3)(x_1^2 + x_3^3) + 3x_3^2 \exp(x_2 x_3)$$

Partial Derivatives of Higher Orders

We can also compute partial derivatives of higher orders. Given a function $f(x_1, x_2)$, its second order partial derivatives are:

$$f_{x_1x_1}(x_1, x_2) = \frac{\partial}{\partial x_1}\left(\frac{\partial f(x_1, x_2)}{\partial x_1}\right)$$

$$f_{x_2x_2}(x_1, x_2) = \frac{\partial}{\partial x_2}\left(\frac{\partial f(x_1, x_2)}{\partial x_2}\right)$$

$$f_{x_1x_2}(x_1, x_2) = \frac{\partial}{\partial x_2}\left(\frac{\partial f(x_1, x_2)}{\partial x_1}\right)$$

$$f_{x_2x_1}(x_1, x_2) = \frac{\partial}{\partial x_1}\left(\frac{\partial f(x_1, x_2)}{\partial x_2}\right)$$

Partial Derivatives of Higher Orders

Example

Let $f(x_1, x_2) = \sin(x_1) + x_1 \exp(x_2)$. Find the partial derivatives of second order of f.

$$
\begin{aligned}
f_{x_1}(x_1, x_2) &= \\
f_{x_2}(x_1, x_2) &= \\
f_{x_1x_1}(x_1, x_2) &= \\
f_{x_2x_2}(x_1, x_2) &= \\
f_{x_1x_2}(x_1, x_2) &= \\
f_{x_2x_1}(x_1, x_2) &=
\end{aligned}
$$

Remark

The partial derivatives $f_{x_1x_2}$, $f_{x_2x_1}$ of second order are called the **mixed derivatives** of f.

Partial Derivatives of Higher Orders

In the previous example, we found that the two partial derivatives $f_{x_1x_2}$, $f_{x_2x_1}$ were equal. This is not always the case. However, we are able to constitute the following assertion:

Theorem

Assume $f : D \to \mathbb{R}$ ($D \subseteq \mathbb{R}^2$) has mixed partial derivatives $f_{x_1x_2}$, $f_{x_2x_1}$ which are continuous in some open disc centered around $(\bar{x}_1, \bar{x}_2)$. Then

$$f_{x_1x_2}(\bar{x}_1, \bar{x}_2) = f_{x_2x_1}(\bar{x}_1, \bar{x}_2).$$