Limits and Continuity

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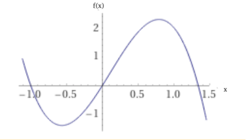
Limits of 1-Variable Functions

Recall the limits of one-variable functions: If

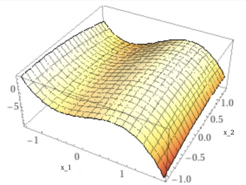
$$\lim_{x \to \bar{x}} f(x) = L,$$

then both side-limits must be equal to L:

$$\lim_{x \to \bar{x}^-} f(x) = \lim_{x \to \bar{x}^+} f(x) = L.$$



For functions of 2 variables, we do not have just two "side-limits", because (x_1, x_2) can approach $\bar{x}_1, \bar{x}_2$ along many directions. When $\lim_{(x_1,x_2) \to (\bar{x}_1,\bar{x}_2)} f(x_1, x_2) = L$, then $f(x_1, x_2)$ will approach the value L as (x_1, x_2) tends to $(\bar{x}_1, \bar{x}_2)$ along **any** direction. From this, we conclude:



Limits of 2-Variable Functions

Lemma

If there exist two different paths C_1, C_2 in the plane such that

$$\lim_{(x_1,x_2)\to(\bar{x}_1,\bar{x}_2),(x_1,x_2)\in C_1} f(x_1,x_2) = L_1$$

and

$$\lim_{(x_1,x_2)\to(\bar{x}_1,\bar{x}_2),(x_1,x_2)\in C_2} f(x_1,x_2) = L_2,$$

where $L_1 \neq L_2$, then $\lim_{(x_1,x_2)\to(\bar{x}_1,\bar{x}_2)} f(x_1,x_2)$ does **not** exist.

Limits of 2-Variable Functions

Example

Prove that the limit

$$\lim_{(x_1,x_2)\to(0,0)} \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2}$$

does **not** exist.

- Points on the x_1-axis have the form $(x_1, x_2) = (x_1, 0)$. For such points, we have

$$\frac{x_1^2 - x_2^2}{x_1^2 + x_2^2} = \frac{x_1^2 - 0^2}{x_1^2 + 0^2} = \frac{x_1^2}{x_1^2} = 1.$$

Thus,

$$\lim_{\substack{(x_1,x_2)\to(0,0)\\ x_2=0}} \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2} = 1.$$

Limits of 2-Variable Functions

- Points on the x_2-axis have the form $(x_1, x_2) = (0, x_2)$. For such points, we have

$$\frac{x_1^2 - x_2^2}{x_1^2 + x_2^2} = \frac{0^2 - x_2^2}{0^2 + x_2^2} = \frac{-x_2^2}{x_2^2} = -1.$$

  Thus

$$\lim_{\substack{(x_1,x_2)\to(0,0) \\ x_1=0}} \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2} = -1.$$

Therefore, the limit

$$\lim_{(x_1,x_2)\to(0,0)} \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2}$$

does not exist.

Limits of 2-Variable Functions

Example

Show that the limit

$$\lim_{(x_1,x_2)\to(0,0)} \frac{4x_1x_2}{x_1x_2 + x_2^3}$$

does **not** exist.

Set

$$f(x_1, x_2) = \frac{4x_1x_2}{x_1x_2 + x_2^3} = \frac{4x_1x_2}{x_2(x_1 + x_2^2)}.$$

The domain of f is

$$D = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0 \text{ and } x_1 + x_2^2 \neq 0\right\}.$$

For any point $(x_1, x_2) \in D$, we have

$$f(x_1, x_2) = \frac{4x_1}{x_1 + x_2^2}.$$

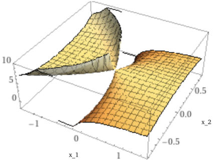
Now we consider points (x_1, x_2) on the parabola $x_1 = ax_2^2$ $(a \neq -1)$, i.e. points of the form (ax_2^2, x_2).

Limits of 2-Variable Functions

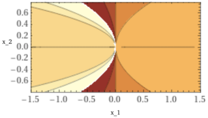
Then

$$f(ax_2^2, x_2) = \frac{4ax_2^2}{ax_2^2 + x_2^2} = \frac{4ax_2^2}{x_2^2(a+1)} = \frac{4a}{a+1}.$$

This means that the limit along the parabola $x_1 = ax_2^2$ $(a \neq -1)$ will depend on the choice of a (and is thus different for different choices of a). Hence, the limit $\lim_{(x_1,x_2)\to(0,0)} \frac{4x_1x_2}{x_1x_2+x_2^3}$ does not exist.



Contour plot



Limits of 2-Variable Functions

To show that a limit does not exist, we must identify two paths along which the limits differ. To show that a limit exists, we cannot use paths, since the limits along all possible paths must be the same and there is no way to check all possible paths. Accordingly, to show that a limit exists, we proceed as in the one-dimensional case: We combine the formal definition of limits and the limit laws to compute limits.

Continuity of a Function

Definition: Continuous Function

A function $f(x_1, x_2)$ is called **continuous** at the point $(\bar{x}_1, \bar{x}_2)$ in its domain if

$$\lim_{(x_1,x_2)\to(\bar{x}_1,\bar{x}_2)} f(x_1, x_2) = f(\bar{x}_1, \bar{x}_2),$$

i.e., the limit of f when $(x_1, x_2) \to (\bar{x}_1, \bar{x}_2)$ exists and is equal to $f(\bar{x}_1, \bar{x}_2)$.

Continuity of a Function

Example

Show that the function $f(x_1, x_2) = 2 + x_1^2 + x_2^2$ is continuous at the point $(0, 0)$.

$$\begin{aligned}
\lim_{(x_1,x_2)\to(0,0)} f(x_1, x_2) &= \lim_{(x_1,x_2)\to(0,0)} (2 + x_1^2 + x_2^2) \\
&= 2 + \lim_{(x_1,x_2)\to(0,0)} x_1^2 + \lim_{(x_1,x_2)\to(0,0)} x_2^2 \\
&= 2 + 0^2 + 0^2 \\
&= 2 \\
&= f(0, 0).
\end{aligned}$$

Hence, f is continuous at $(0, 0)$.

Continuity of a Function

Example

Let

$$f(x_1, x_2) = \begin{cases} \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2} & (x_1, x_2) \neq (0, 0) \\ 0 & (x_1, x_2) = (0, 0). \end{cases}$$

Is f continuous at the point $(0, 0)$?

Here, $f(0, 0) = 0$, but we have shown earlier that $\lim_{(x_1, x_2) \to (0,0)} f(x_1, x_2)$ does not exist. Therefore, f is not continuous at $(0, 0)$.