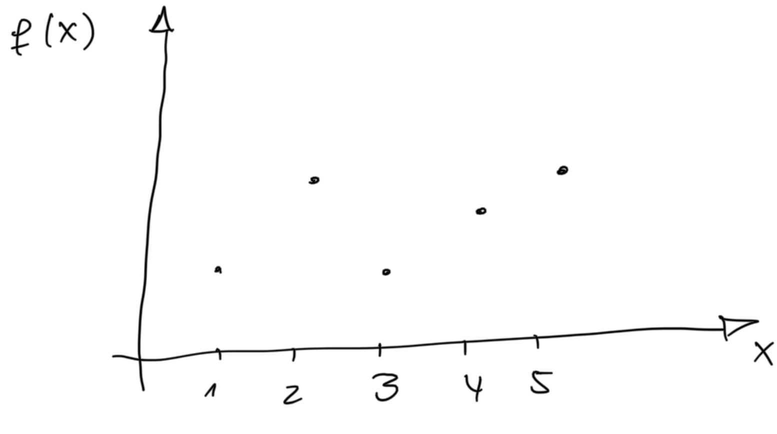
Lecture 18

Multivariable Calculus

So far, we have only encountered functions that depend on one variable, that is, functions

$$f : \underline{\underline{D}} \to \mathbb{R} \,, \quad \text{where } D \underbrace{\subseteq}_{\text{"subset"}} \mathbb{R} .$$

Every number $x \in D$ $(D \subseteq \mathbb{R})$ is mapped to a real number $f(x)$:



The set $D \subseteq \mathbb{R}$ is called the domain of f, while the set

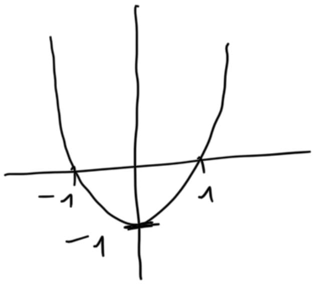
$$\{ f(x) : x \in D \} \subseteq \mathbb{R}$$

(the set of all possible values of f) is called the range of f.

Example: The function $f(x) = x^2 - 1$ $(x \in \mathbb{R})$ maps every real value $x \in \mathbb{R}$

$(x \in \mathbb{R})$ ...

to $x^2 - 1$.



The domain of f is $\mathbb{R}$, and the range of f is $[-1, +\infty)$.

We often encounter functions of multiple variables, i.e., functions

$$f : \underline{D} \to \mathbb{R}, \quad D \subseteq \mathbb{R}^n,$$

where $\mathbb{R}^n := \{ (x_1, x_2, \dots, x_n) : x_1, x_2, \dots, x_n \in \mathbb{R}$

The set $D \subseteq \mathbb{R}^n$ is called the <u>domain</u> of f, while the set

$$\{ f(x_1, x_2, \dots, x_n) : (x_1, x_2, \dots, x_n) \in D \}$$

is called the <u>range</u> of f.

<u>Example</u>: Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$, defined as

$$f(x_1, x_2, x_3) = x_1 - x_3 + e^{x_1 \cdot x_2 \cdot x_3}.$$

Every vector $(x_1, x_2, x_3)^T \in \mathbb{R}^3$ is mapped to the real number

$$x_1 - x_3 + e^{x_1 x_2 x_3} \quad (\in \mathbb{R}).$$

For example,

$$f(3,2,0) = 3 - 0 + e^{3\cdot 2\cdot 0}$$
$$= 3 + \underbrace{e^{0}}_{=1} = \underline{\underline{4}}$$

<u>Example</u>: What is the maximal domain of $f(x_1, x_2, x_3) = \dfrac{x_1 \cdot x_2}{x_3^2}$?

Evaluate the function f at the points $(2,3,-1)$ and $(-1,2,3)$.

- In order for f to be well-defined, we need $x_3^2 \neq 0 \iff x_3 \neq 0$.
  So, the maximal domain of f is
  $$D = \{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 \neq 0 \}$$
- $f(2,3,-1) = \dfrac{2 \cdot 3}{(-1)^2} = \underline{\underline{6}}$
- $f(-1,2,3) = \dfrac{-1 \cdot 2}{3^2} = \underline{\underline{-\dfrac{2}{9}}}$

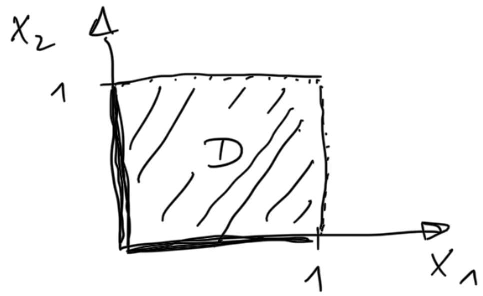
<u>Example</u>: Let $D = \{ (x_1, x_2) \in \mathbb{R}^2 : 0 \leq x_1 \leq 1, \; 0 \leq x_2 \leq 1 \}$

and consider the function

$$f : D \to \mathbb{R}, \quad \underline{f(x_1, x_2) = x_1 + x_2}.$$

Sketch the domain of f in

the $x_1 x_2$ - plane and find the range of f.



We find the range of f:

If $(x_1, x_2) \in D$, then

$$\left.\begin{array}{l} 0 \le x_1 \le 1 \\ 0 \le x_2 \le 1 \end{array}\right\} \quad 0 \le x_1 + x_2 \le 2,$$

hence, the range of f is a subset of $[0,2]$.

Every number $y \in [0,2]$ can be written as a value of f; for example, we can write

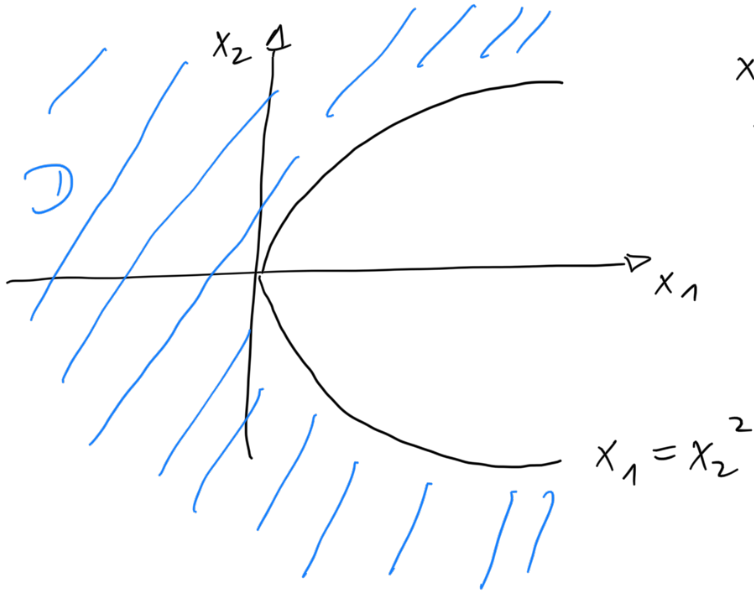
$$y = \frac{y}{2} + \frac{y}{2} = f\left(\frac{y}{2}, \frac{y}{2}\right).$$

Hence, the range of f is $[0,2]$.

Example: Find and sketch the maximal domain of $f(x_1, x_2) = \sqrt{x_2^2 - x_1}$.

In order for $\sqrt{x_2^2 - x_1}$ to be well-defined on $\mathbb{R}$, we need $x_2^2 - x_1 \geq 0$.

Therefore, the maximal domain of f is $D = \{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \le x_2^2 \}$



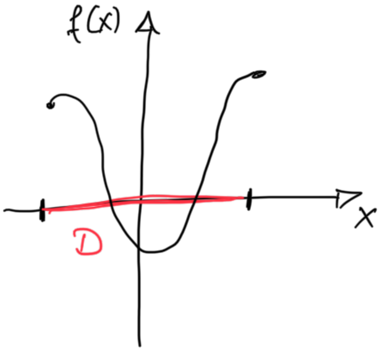
$x_1 = x_2^2$

$\Rightarrow x_2 = \sqrt{x_1}$

or $x_2 = -\sqrt{x_1}$

Functions of one variable have a domain $D \subseteq \mathbb{R}$ and their graph is the set

$$G_f = \{ (x, y) \in \mathbb{R}^2 : x \in D \text{ and } y = f(x) \}$$



The graph of a function $f : D \to \mathbb{R}$ is a subset of the two-dimensional plane.

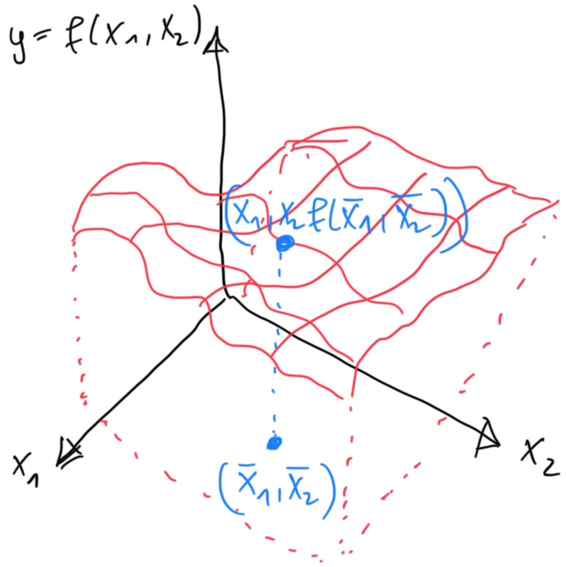
We can plot the graph of a function of two variables. For a function

$$f : D \to \mathbb{R}, \quad D \subseteq \mathbb{R}^2,$$

we define the graph of f to be the set

$$G_f = \{ (x_1, x_2, y) \in \mathbb{R}^3 : (x_1, x_2) \in D,$$

$- f -$ $\qquad$ $y = f(x_1, x_2)\}$.



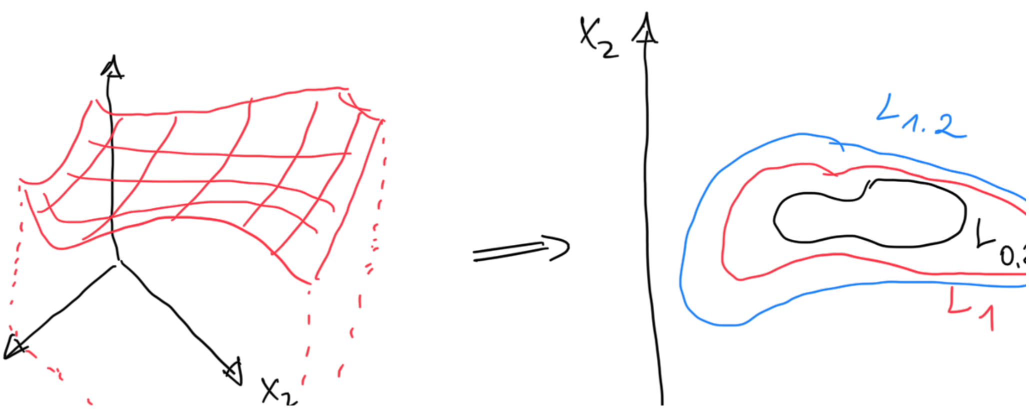
The graph of the function $f : D \to \mathbb{R}$ $(D \subseteq \mathbb{R}^2)$ is a surface in the three-dimensional space.

Let $f : D \to \mathbb{R}$ $(D \subseteq \mathbb{R}^2)$ be a function of two variables. We define <u>level sets</u> of f to be the sets of the form

$$\boxed{L_c = \{(x_1, x_2) \in D \ : \ \underline{f(x_1, x_2) = c}\}}$$

for some value $c \in \mathbb{R}$.

In other words, level sets of f are the sets of points (x_1, x_2) in the domain of f, where f is constant and equal to some value $c \in \mathbb{R}$.



$L_{0,8} = \{ (x_1,x_2) \in D : f(x_1,x_2) =$

$L_1 = \{ (x_1,x_2) \in D : f(x_1,x_2) = 1 \}$

$L_{1.2} = \{ (x_1,x_2) \in D : f(x_1,x_2) = 1.$

<u>Example</u>: let $f(x_1, x_2) = (x_1 + 2x_2 - 1)^3$.
Find the level sets of f.

- $f(x_1, x_2) = c \iff (x_1 + 2x_2 - 1)^3 = c$
  - For $c > 0$, this is equivalent to
    $x_1 + 2x_2 - 1 = \sqrt[3]{c}$
    $\iff \underline{x_1 + 2x_2} = 1 + \sqrt[3]{c}$
  - For $c \leq 0$:
    $x_1 + 2x_2 - 1 = -\sqrt[3]{c}$
    $\iff \underline{x_1 + 2x_2} = 1 - \sqrt[3]{c}$.

So, the level sets $L_c = \{(x_1, x_2) \in \mathbb{R}^2 : f(x_1, x_2) = c\}$ are lines in the plane.

<u>Example</u>: Let $D = \{ (x_1,x_2) \in \mathbb{R}^2 : \underline{x_1^2 + x_2^2 \leq 4} \}$

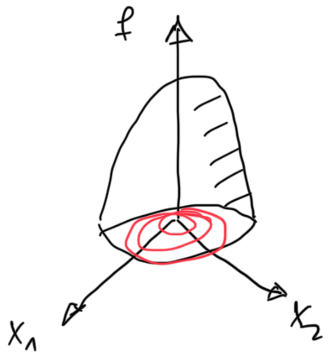
Compare the level sets of
$f : D \to \mathbb{R}, \quad f(x_1, x_2) = 4 - x_1^2 - x_2^2$
$g : D \to \mathbb{R}, \quad g(x_1, x_2) = \sqrt{4 - x_1^2 - x_2^2}$

- We have:

$$f(x_1, x_2) = c \iff 4 - x_1^2 - x_2^2 = c$$

$$\iff \underline{x_1^2 + x_2^2} = 4 - c$$

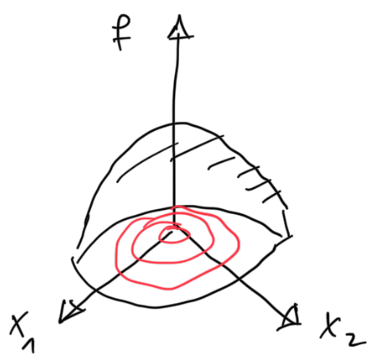
Thus, the level set L_c of f is a circle with center $(0,0)$ and radius $\sqrt{4-c}$.



- $$g(x_1, x_2) = c \iff \sqrt{4 - x_1^2 - x_2^2} = c$$

$$\iff x_1^2 + x_2^2 = 4 - c^2.$$

Thus, the level set L_c of g is a circle with center at $(0,0)$ and radius $\sqrt{4-c^2}$



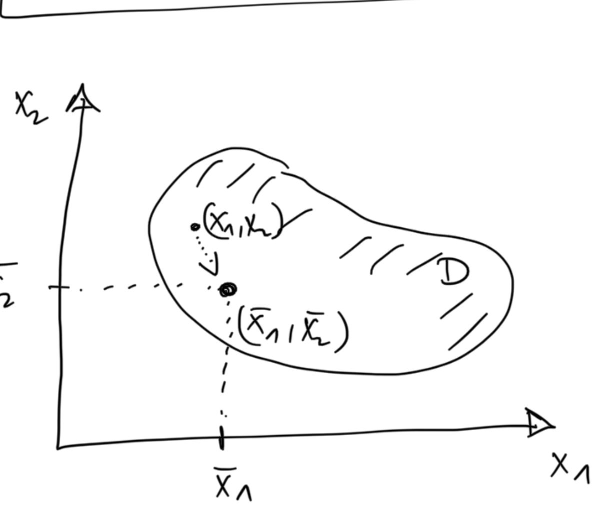
If, for example, $T(x_1, x_2)$ denotes the temperature at the point (x_1, x_2), then the level sets L_c are sets on the plane whose points have equal temperature.

Limits and Continuity of 2-Variable Functions

Let $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^2$, be a function and $(\bar{x}_1, \bar{x}_2) \in D$ is a point in D.

Suppose that when the point $(x_1, x_2) \in D$ gets close to $(\bar{x}_1, \bar{x}_2)$ (but is always different from $(\bar{x}_1, \bar{x}_2)$), the value $f(x_1, x_2)$ becomes arbitrarily close to some real number L. Then we say that the limit of the function f when (x_1, x_2) tends to $(\bar{x}_1, \bar{x}_2)$ is equal to L, and we write

$$\boxed{\lim_{(x_1,x_2) \to (\bar{x}_1, \bar{x}_2)} f(x_1, x_2) = L.}$$



For example, let $f(x_1, x_2) = x_1 + x_2^2$.
When $(x_1, x_2) \to (1, 2)$, the values of f become arbitrarily close to the number $1 + 2^2 = 5$. So, $\lim\limits_{(x_1,x_2)\to(1,2)} (x_1 + x_2^2) = 5$.

When we need to calculate limits, we can use the following rules:
Suppose that $a \in \mathbb{R}$, $(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2$, and

$$\lim_{(x_1,x_2)\to(\bar{x}_1,\bar{x}_2)} f(x_1, x_2) = L_1 \quad \text{and}$$

$$\lim_{(x_1,x_2)\to(\bar{x}_1,\bar{x}_2)} g(x_1, x_2) = L_2$$

$(L_1, L_2 \in \mathbb{R})$.

Then

- $\lim\limits_{(x_1,x_2)\to(\bar{x}_1,\bar{x}_2)} [f(x_1, x_2) + g(x_1,x_2)] = L_1 + L_2$.
- $\lim\limits_{(x_1,x_2)\to(\bar{x}_1,\bar{x}_2)} a \cdot f(x_1, x_2) = a \cdot L_1$
- $\lim\limits_{(x_1,x_2)\to(\bar{x}_1,\bar{x}_2)} f(x_1, x_2) \cdot g(x_1, x_2) = L_1 \cdot L_2$
- $f(x_1, x_2)$ $\quad L_1 \quad$ (if $L_2 \neq 0$)

- $\lim\limits_{(x_1,x_2)\to(\bar{x}_1,\bar{x}_2)} \frac{f(x_1,x_2)}{g(x_1,x_2)} = \frac{L_1}{L_2}$

<u>Examples:</u>

(a) $\lim\limits_{(x_1,x_2)\to(0,0)} \left(\underline{x_1^2 + x_2^2}\right) = \lim\limits_{\substack{(x_1,x_2)\\ \to(0,0)}} x_1^2 + \lim\limits_{\substack{(x_1,x_2)\\ \to(0,0)}} x_2^2$

$= \underline{\underline{0}}$

(b) $\lim\limits_{(x_1,x_2)\to(4,-3)} \left(x_1^2 + x_2^2\right) = \lim\limits_{\substack{(x_1,x_2)\\ \to(4,-3)}} x_1^2 + \lim\limits_{(x_1,x_2)\to(4,-3)} x_2^2$

$= 4^2 + (-3)^2$

$= \underline{\underline{25}}$

(c) $\lim\limits_{(x_1,x_2)\to(-1,2)} x_1^2 \cdot x_2 = (-1)^2 \cdot 2 = \underline{\underline{2}}$

(d) $\lim\limits_{(x_1,x_2)\to(1,3)} x_1^2 x_2 + 3x_1 = 1^2 \cdot 3 + 3 \cdot 1$

$= \underline{\underline{6}}$

(e) $\lim\limits_{\substack{(x_1,x_2)\\ \to(-1,3)}} \frac{3x_1}{x_2} = \frac{3\cdot(-1)}{3} = \underline{\underline{-1}}$

(f) $\lim \frac{4x_2 + 2x_1}{x_1^2 + 2x_1x_2 - 3} = \frac{4\cdot 0 + 2\cdot 2}{2^2 + 2\cdot 2\cdot 0 - 3}$

(x_1, x_2) $\quad x_1^2 + \alpha x_1 x_2$

$\rightarrow (2,0)$ $\quad = \frac{4}{1} = \underline{\underline{4}}$