Planes in the 3-Dimensional Space

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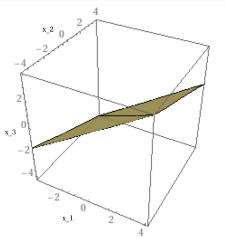
The standard equation of a plane in the 3-dimensional space is

$$Ax_1 + Bx_2 + Cx_3 + D = 0, \quad \text{where } |A| + |B| + |C| > 0.$$

For example, the equation

$$2x_1 - x_2 - 3x_3 = 1$$

describes a 3-dimensional plane:



One point on this plane is, for example, $(1, -2, 1)$, as its coordinates satisfy the equation.

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Any 3-dimensional plane is uniquely determined by a point on it and a vector perpendicular to it.
Assume the point $\bar{x} = (\bar{x}_1, \bar{x}_2, \bar{x}_3)^T$ belongs to the plane and the vector $n = (n_1, n_2, n_3)^T$ is perpendicular to the plane. Denote by $x = (x_1, x_2, x_3)^T$ be an arbitrary vector on the plane. The vector $x - \bar{x}$ is perpendicular to n, and therefore

$$(x - \bar{x})^T n = 0.$$

This is the **vector equation** of a plane.
Expanding this scalar product, we get

$$(x_1 - \bar{x}_1)n_1 + (x_2 - \bar{x}_2)n_2 + (x_3 - \bar{x}_3)n_3 = 0.$$

This is the **Cartesian equation** of a plane.

Planes in the 3-Dimensional Space: Example

Find the equation of the plane in the 3-dimensional space that goes through the point $(2, 0, 3)$ and is perpendicular to the vector $(-1, 4, 1)^T$.

Solution: We have $\bar{x} = (2, 0, 3)^T$ and $n = (-1, 4, 1)^T$. Therefore:

$$
\begin{aligned}
&(x_1 - \bar{x}_1)n_1 + (x_2 - \bar{x}_2)n_2 + (x_3 - \bar{x}_3)n_3 \\
&= (x_1 - 2) \cdot (-1) + (x_2 - 0) \cdot 4 + (x_3 - 3) \cdot 1 \\
&= -x_1 + 2 + 4x_2 + x_3 - 3 \\
&= -x_1 + 4x_2 + x_3 - 1 = 0.
\end{aligned}
$$