Equations in Parametric Form

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Equations in Parametric Form

Apart from the Cartesian equation and the vector equation, a line can be described by equations in parametric form (i.e., equations that involve some parameters).

Example

Given the equations

$$
\begin{aligned}
x &= 2t - 1 \\
y &= 3 - t, \quad t \in \mathbb{R},
\end{aligned}
$$

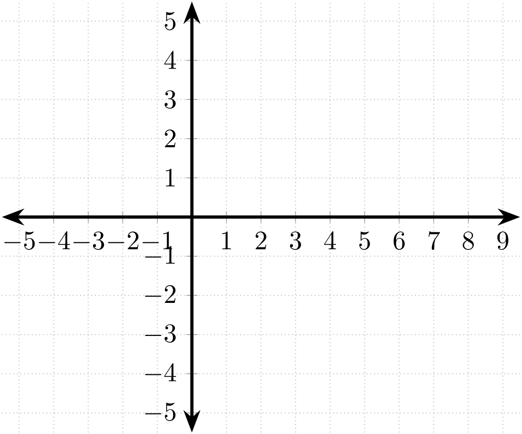
where does the point (x, y) lie when x, y satisfy these equations?
Let us eliminate the parameter t from the equations:

$$
\begin{aligned}
&y = 3 - t \quad \Longrightarrow \quad t = 3 - y \\
&x = 2t - 1 \quad \Longrightarrow \quad x = 2(3 - y) - 1 \quad \Longrightarrow \quad x + 2y = 5,
\end{aligned}
$$

and this is the equation of a line.

Equations in Parametric Form

For $t = 1$, we have $(1, 2)$, and for $t = 5$, we have $(9, -2)$:



Equations in Parametric Form

Assume now that for a line we only know two facts:

- the line goes through the point $\bar{x} = (\bar{x}_1, \bar{x}_2)^T$
- the line is parallel to the vector $u = (u_1, u_2)^T$.

For the arbitrary point $x = (x_1, x_2)^T$ on the line, the vector $x - \bar{x}$ will be parallel to u, hence, there exists a $t \in \mathbb{R}$ such that

$$x - \bar{x} = tu \quad \Longrightarrow \quad \begin{pmatrix} x_1 - \bar{x}_1 \\ x_2 - \bar{x}_2 \end{pmatrix} = t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

This implies

$$\begin{aligned} x_1 - \bar{x}_1 &= u_1 t \\ x_2 - \bar{x}_2 &= u_2 t, \quad t \in \mathbb{R}. \end{aligned}$$

Hence,

$$\begin{aligned} x_1 &= u_1 t + \bar{x}_1 \\ x_2 &= u_2 t + \bar{x}_2, \quad t \in \mathbb{R}. \end{aligned} \tag{1}$$

The pair of equations (1) is called **parametric equation** of the line.

Equations in Parametric Form - Example

Find the parametric equations of the line in the plane that goes through the point $(2, 1)$ in the direction of the vector $(-1, -3)^T$. What is the standard (Cartesian) equation of the same line?

Solution: Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\bar{x} = \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $u = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$. Then

$$x - \bar{x} = tu \quad \Longrightarrow \quad \begin{pmatrix} x_1 - 2 \\ x_2 - 1 \end{pmatrix} = t \begin{pmatrix} -1 \\ -3 \end{pmatrix}.$$

This implies

$$\begin{aligned} x_1 &= 2 - t \\ x_2 &= 1 - 3t, \quad t \in \mathbb{R}. \end{aligned}$$

To find the standard equation of the line, we eliminate $t \in \mathbb{R}$ from the parametric equations:

$$x_1 = 2 - t \quad \Longrightarrow \quad t = 2 - x_1$$

and

$$x_2 = 1 - 3t \quad \Longrightarrow \quad x_2 = 1 - 3(2 - x_1) \quad \Longrightarrow \quad \boxed{x_2 = 3x_1 - 5}.$$

Parametric Equation: Example

Find the parametric equations of the line in the plane that goes through the points $(-1,2)^T$ and $(3,5)^T$.

Solution: The line goes through the point $(-1,2)^T$ and is parallel to the vector

$$u = \begin{pmatrix} 3 \\ 5 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

Hence, the line in parametric equations is

$$\begin{aligned} x_1 &= 4t - 1 \\ x_2 &= 3t + 2, \quad t \in \mathbb{R}. \end{aligned}$$

Parametric Equation: Example

Formulate the parametric equations of the line $2x_1 - 3x_2 + 1 = 0$.

Solution: We set one of the variables x_1, x_2 equal to the parameter $t \in \mathbb{R}$, for example, let $x_1 = t$, $t \in \mathbb{R}$. Then

$$\begin{aligned} 2x_1 - 3x_2 + 1 = 0 \quad &\Longrightarrow \quad 3x_2 = 2t + 1 \\ &\Longrightarrow \quad x_2 = \frac{2}{3}t + \frac{1}{3}. \end{aligned}$$

Hence, the parametric equations of the line are

$$\begin{aligned} x_1 &= t \\ x_2 &= \frac{2}{3}t + \frac{1}{3}, \quad t \in \mathbb{R}. \end{aligned}$$

Remark

This solution we found is not unique. For example, one could have set $x_2 = t$ and find

$$\begin{aligned} x_1 &= \frac{3}{2}t - \frac{1}{2} \\ x_2 &= t, \quad t \in \mathbb{R}. \end{aligned}$$