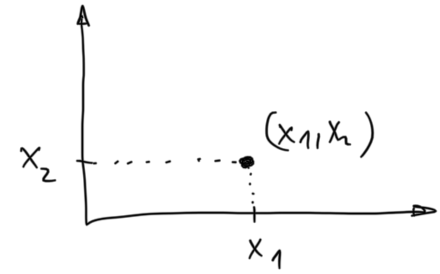
Lecture 16

Analytic Geometry

We have already seen that the 2-dimensional plane can be identified with the set of ordered pairs $(x_1, x_2) \in \mathbb{R}^2$ ($x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$).



Generally, for any $n \geq 1$, $n \in \mathbb{N}$, we define

$$\mathbb{R}^n := \{ (x_1, \dots, x_n) : x_1, \dots, x_n \in \mathbb{R} \}.$$

Elements of $\mathbb{R}^n$ are represented as n-tupels $(x_1, \dots, x_n)$ or n-dim. column vectors

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (x_1, \dots, x_n)^T,$$

where "T" stands for "transpose" and means the following: For a matrix

$$A = (a_{ij})_{\substack{i \leq m \\ j \leq n}},$$

its transpose is

$$A^T = (a_{ji})_{\substack{j \leq n \\ i \leq m}}.$$

E.g.: $A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 5 \end{pmatrix} \Rightarrow A^T = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 3 & 5 \end{pmatrix}.$

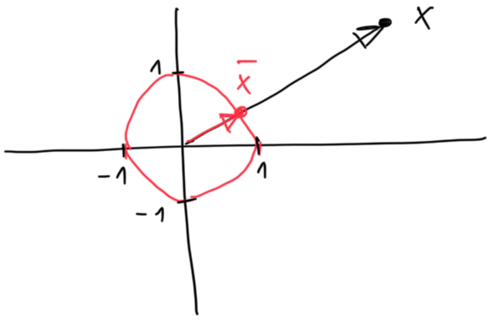
- A vector $x \in \mathbb{R}^n$ is called the <u>unit vector</u> if

$$|x| = 1.$$

- Whenever we are given a vector $x \in \mathbb{R}^n$, $x \neq 0$, we can <u>normalize</u> it in order to obtain a <u>unit</u> vector in the same direction as x: We multiply x by the number $\frac{1}{|x|}$ and obtain the new vector

$$\boxed{\hat{x} = \frac{x}{|x|}}$$

which has length 1.



E.g.: Normalize the vector $x = \begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix}$.

$$\Rightarrow |x| = \sqrt{3^2 + (-6)^2 + 6^2} = \sqrt{81} = \underline{\underline{9}}$$

$$\hat{x} = \frac{1}{9} \cdot \begin{pmatrix} 3 \\ -6 \\ 6 \end{pmatrix} = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)^T.$$

Verify that $\hat{x}$ is indeed a unit vector:

$$|\hat{x}| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^2}$$

$$= \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = \sqrt{\frac{9}{9}} = \sqrt{1} = \underline{\underline{1}}$$

- Scalar Product / Dot Product :

$$x \cdot y = \underbrace{x^T y}_{(x_1, \dots, x_n) \cdot \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}} = x_1 y_1 + \dots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

E.g.: Given $x = (2, 3, 1)^T$, $y = (-1, 2, 0)^T$, we have

$$x \cdot y = x^T y = (2, 3, 1) \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

$$= 2 \cdot (-1) + 3 \cdot 2 + 1 \cdot 0$$

$$= \underline{\underline{4}}$$

- The angle between two vectors

$$\boxed{x \cdot y = |x| \cdot |y| \cdot \cos \theta}$$

$$\Rightarrow \cos \theta = \frac{x \cdot y}{|x| \cdot |y|} \Rightarrow \underbrace{\theta = \arccos\left(\frac{x \cdot y}{|x| \cdot |y|}\right)}_{\text{angle between } x \text{ and } y}$$

E.g.: Angle between $x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$:

$$x \cdot y = 2 \cdot 1 + 1 \cdot 1 = 3$$

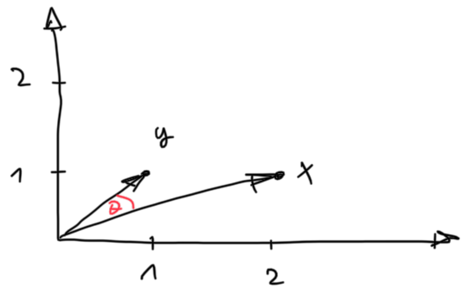
$$|x| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

$$|y| = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\Rightarrow \cos\theta = \frac{x\cdot y}{|x|\cdot|y|} = \underline{\frac{3}{\sqrt{5}\cdot\sqrt{2}}}$$

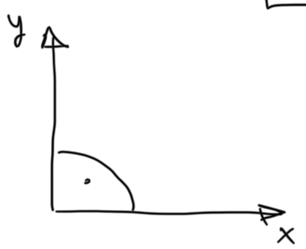
$$\Rightarrow \theta = \arccos\frac{3}{\sqrt{5}\sqrt{2}} = \underline{0.3218 \text{ radians}}$$

$(\sim 18.4°)$



- Two vectors x, y are <u>perpendicular</u> if

$$\boxed{x\cdot y = 0}$$



- <u>Example</u>: Find the value of $a \in \mathbb{R}$ s.t.

$$x = \begin{pmatrix}1\\a\\-1\end{pmatrix} \text{ is perpendicular to}$$

$$y = \begin{pmatrix}2\\1\\3\end{pmatrix}.$$

$$\Rightarrow x\cdot y = 1\cdot 2 + a\cdot 1 + (-1)\cdot 3$$
$$= 2 + a - 3$$
$$= a - 1$$

$\Rightarrow$ the vectors are perpendicular if

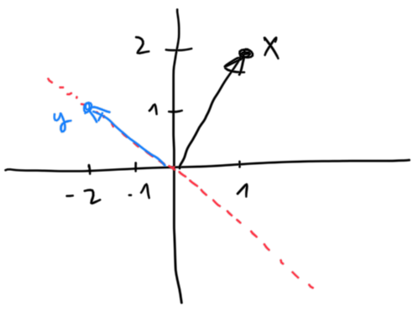
$$\underline{\underline{a = 1}}.$$

- <u>Example</u>: Let $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Find a vector $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ s.t. x and y are perpendicular.

$\Rightarrow \quad x \cdot y = 0 = 1 \cdot y_1 + 2 y_2$

$\rightarrow \quad y_1 = -2 y_2$.

For example, $y = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ is perpendicular to x.



- <u>Lines in the Plane</u>

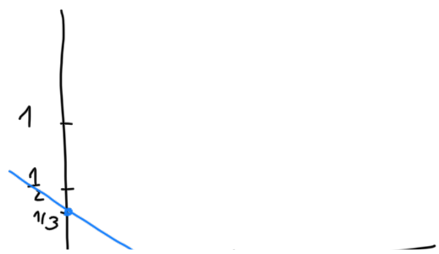
There are several ways to write the equation of a line in the plane:

- Cartesian equation
- vector equation
- parametric equations.

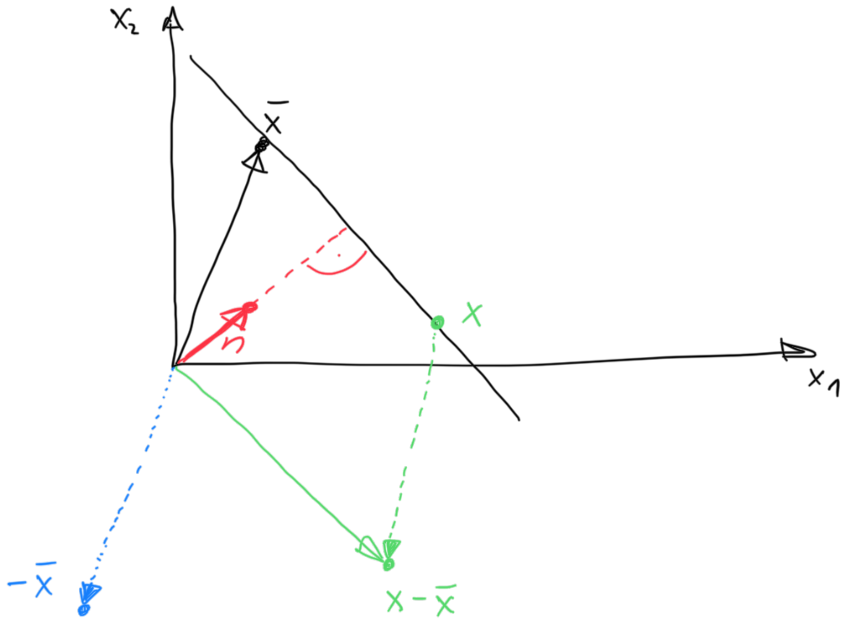
- The <u>Cartesian equation</u> of a line in the plane:

$$\boxed{A x_1 + B x_2 + C = 0}, \quad A \neq 0 \text{ or } B \neq 0.$$

E.g.: $2x_1 + 3x_2 - 1 = 0$



- Suppose we know two things about the line:
  - it goes through the point $\bar{x} = \begin{pmatrix}\bar{x}_1 \\ \bar{x}_2\end{pmatrix}$
  - it is perpendicular to the given vector n



Let $x = \begin{pmatrix}x_1 \\ x_2\end{pmatrix}$ be an arbitrary point on the line.

Then the vector $\overbrace{x - \bar{x}}^{= x + (-\bar{x})}$ is parallel to the line, and hence perpendicular to n.

Therefore:

$$\boxed{(x - \bar{x}) \cdot n = 0}$$

(vector equation of a line)

E.g.: Write the vector equation of the line that goes through $\begin{pmatrix}4 \\ 3\end{pmatrix}$ and is perpendicular to $n = \begin{pmatrix}1 \\ 2\end{pmatrix}$.

What is the Cartesian eq. of the line?

- Vector eq.: $\bar{x} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $n = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$\Rightarrow (x - \bar{x}) n = \left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \end{pmatrix} \right) \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0 .$$

- Cartesian eq.:

$$(x_1 - 4) \cdot 1 + (x_2 - 3) \cdot 2$$
$$= x_1 - 4 + 2x_2 - 6$$
$$= \underline{\underline{x_1 + 2x_2 - 10 = 0}} .$$