Leslie Matrices Revisited

An Application of Eigenvectors and Eigenvalues

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Leslie Matrix

Recall that if $L = \begin{pmatrix} F_0 & F_1 & \dots & F_{m-1} \\ P_0 & 0 & \dots & 0 \\ 0 & P_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$ is the Leslie matrix of a population, then:

- the females are split into m age groups
- F_i is the offspring-rate of the i-th age group $(i = 0, \dots, m-1)$
- P_i is the survival rate of the i-th age group $(i = 0, \dots, m-1)$.

Leslie Model: Example

Suppose the Leslie matrix is $L = \begin{pmatrix} 1.5 & 2 \\ 0.08 & 0 \end{pmatrix}$. This means that we have two age groups: 0-year olds and 1-year olds. The maximum age is one year. 0-year olds produce on average 1.5 surviving female offspring, and 1-year olds produce on average 2 surviving female offspring. The number 0.08 describes the survival probability of the 0-year olds, that is, 8% of the 0-year olds will survive. We start with the population vector

$$N(0) = \begin{pmatrix} 100 \\ 100 \end{pmatrix},$$

that means that at time $t = 0$ there are 100 0-year olds and 100 1-year olds. We can use the relation

$$N(t+1) = LN(t)$$

to find the successive population vectors. These are

$$\begin{pmatrix} 100 \\ 100 \end{pmatrix}, \begin{pmatrix} 350 \\ 8 \end{pmatrix}, \begin{pmatrix} 541 \\ 28 \end{pmatrix}, \begin{pmatrix} 868 \\ 43 \end{pmatrix}, \begin{pmatrix} 1388 \\ 69 \end{pmatrix}, \begin{pmatrix} 2221 \\ 111 \end{pmatrix}, \dots$$

Leslie Model: Example

Let us look at the successive ratios of 0-year olds, i.e.

$$q_0(t) = \frac{N_0(t)}{N_0(t-1)}, \quad t = 1, 2, \ldots$$

These are

$$3.5, \quad 1.55, \quad 1.60, \quad 1.5991, \quad 1.6001, \quad 1.5997, \quad 1.6001, \quad \ldots$$

These ratios seem to converge to a specific number (namely, 1.6) when $t \to \infty$.

What about the corresponding quotients of the 1-year olds,

$$q_1(t) = \frac{N_1(t)}{N_1(t-1)}, \quad t = 1, 2, \ldots ?$$

We have the ratios

$$0.08, \quad 3.5, \quad 1.536, \quad 1.605, \quad 1.609, \quad 1.604, \quad \ldots$$

and it seems that these quotients converge to the same number.

Leslie Model: Example

We now look at the proportion of the 0-year olds in the total population, namely

$$p_0(t) = \frac{N_0(t)}{N_0(t) + N_1(t)}, \quad t = 0, 1, 2, \ldots$$

Its values are

$$0.5, \;\; 0.9777, \;\; 0.9528, \;\; 0.9526, \;\; \ldots$$

and again it looks like these percentages converge to a number (95.2%). This means that asymptotically, the population will tend to consist of

$$\approx 95.2\% \quad \text{0-year olds and}$$
$$\approx 4.8\% \quad \text{1-year olds.}$$

In other words, these observations suggest that the population vectors tend to become multiples of one vector $N(t)$ for which

$$N(t+1) = LN(t) = \lambda N(t), \quad \text{where } \lambda = 1.6.$$

Leslie Model: Stable Age Distribution

$$N(t+1) = LN(t) = \lambda N(t), \quad \text{where } \lambda = 1.6.$$

This means that the population of 0- and 1-year olds at time $t+1$ is a multiple of the population of 0- and 1-year olds at time t.
Mathematically, we know that $N(t)$ is an eigenvector for L. This vector is called a **stable age distribution** vector.

Note that $N(t)$ does not need to be an eigenvector for any t. For example, $N(0) = \begin{pmatrix} 100 \\ 100 \end{pmatrix}$ is not a stable age distribution, because

$$N(1) = LN(0) = \begin{pmatrix} 350 \\ 8 \end{pmatrix} \neq \lambda N(0) = \lambda \begin{pmatrix} 100 \\ 100 \end{pmatrix}.$$

Leslie Model: Example - Stable Age Distribution

However, any vector where the females in age class 0 are about 95.2% of the population is a stable age distribution, for example the vector $N(0) = \begin{pmatrix} 2000 \\ 100 \end{pmatrix}$:

$$N(1) = LN(0) = \begin{pmatrix} 1.5 & 2 \\ 0.08 & 0 \end{pmatrix} \begin{pmatrix} 2000 \\ 100 \end{pmatrix} = \begin{pmatrix} 3200 \\ 160 \end{pmatrix} = \lambda N(0) = \lambda \begin{pmatrix} 2000 \\ 100 \end{pmatrix}$$

with $\lambda = 1.6$.

Leslie Model: Growth Parameter

In terms of the Leslie matrix, the largest eigenvalue is interpreted as the **growth parameter**, that is, it determines how the population grows. Its associated eigenvector is a stable age distribution.

Stable Age Distribution

How do we find the stable age distribution in general?

For $L = \begin{pmatrix} 1.5 & 2 \\ 0.08 & 0 \end{pmatrix}$, as in our example, we find the eigenvalues and eigenvectors:

$$\begin{aligned} \det(L - \lambda I) &= \begin{vmatrix} 1.5 - \lambda & 2 \\ 0.08 & -\lambda \end{vmatrix} \\ &= -\lambda(1.5 - \lambda) - 0.16 \\ &= \lambda^2 - 1.5\lambda - 0.16. \end{aligned}$$

We solve

$$\det(L - \lambda I) = 0 \iff \lambda^2 - 1.5\lambda - 0.16 = 0.$$

This gives $\lambda_1 = \frac{8}{5} = 1.6$ and $\lambda_2 = -\frac{1}{10} = -0.1$.

Stable Age Distribution

We now find the corresponding eigenvectors.

For $\lambda_1 = 1.6$, we obtain:

$$Lx = 1.6x \quad \Longrightarrow \quad \begin{pmatrix} 1.5 & 2 \\ 0.08 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1.6x_1 \\ 1.6x_2 \end{pmatrix}.$$

This gives the two equations

$$\begin{aligned} 1.5x_1 - 2x_2 &= 1.6x_1 \\ 0.08x_2 &= 1.6x_2 \\ \Longrightarrow \quad x_1 &= 20x_2. \end{aligned}$$

For example, $x = \begin{pmatrix} 20 \\ 1 \end{pmatrix}$ is an eigenvector of L associated to $\lambda_1 = 1.6$.

For $\lambda_2 = -0.1$, we get

$$Ly = -0.1y \quad \Longrightarrow \quad \begin{pmatrix} 1.5 & 2 \\ 0.08 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -0.1y_1 \\ -0.1y_2 \end{pmatrix},$$

giving $1.5y_1 + 2y_2 = -0.1y_1$, and therefore $4y_1 + 5y_2 = 0$. For example, $y = \begin{pmatrix} 5 \\ -4 \end{pmatrix}$ is an eigenvector of L associated to $\lambda_2 = -0.1$.

Example

Let us suppose we start studying the population when $N(0) = \begin{pmatrix} 105 \\ 1 \end{pmatrix}$.

We express this vector as a *linear combination* of the eigenvectors x and y of L:

$$N(0) = \begin{pmatrix} 105 \\ 1 \end{pmatrix} = 5 \cdot \begin{pmatrix} 20 \\ 1 \end{pmatrix} + \begin{pmatrix} 5 \\ -4 \end{pmatrix} = 5x + y.$$

The population vector at time t is

$$\begin{aligned} N(t) &= L^t N(0) \\ &= L^t(5x + y) \\ &= 5L^t x + L^t y \\ &= 5(1.6)^t x + (-0.1)^t y. \end{aligned}$$

Because the eigenvalue $\lambda_1 = 1.6$ is bigger, the first term will dominate in the sum and will largely impact the asymptotic behaviour of $N(t)$.

Summary

Suppose L is a 2×2 Leslie matrix with eigenvalues λ_1 and λ_2.

- The larger eigenvalue is the growth parameter of the population.
- The eigenvector that corresponds to the larger eigenvalue is a stable age distribution.

(The same applies to Leslie matrices of larger dimension, for example 3×3 matrices.)