Lecture 14

Eigenvalues and Eigenvectors

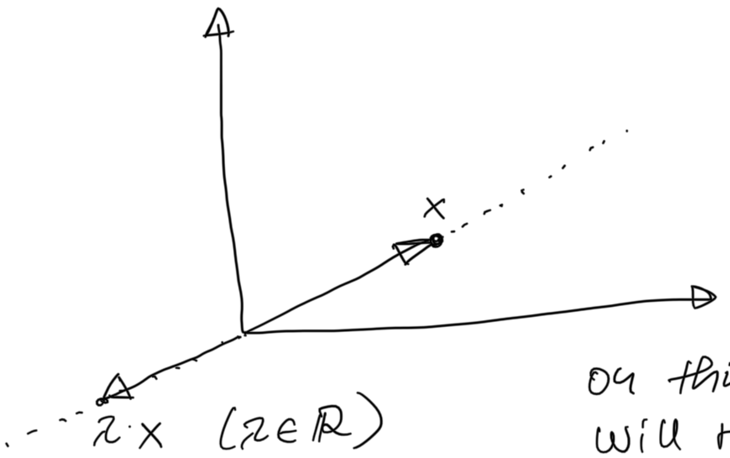
<u>Definition</u>: Assume that A is a square matrix. A <u>nonzero</u> vector x that satisfies the equation

$$\boxed{Ax = \lambda \cdot x}$$

is called an <u>eigenvector</u> of the matrix A, and the number λ is called an <u>eigenvalue</u> of A.

<u>Remark</u>: If we apply A to an eigenvector x (i.e., we compute $A \cdot x$), the result is a multiple of x.

$\Rightarrow$ Geometric interpretation:



If we draw a straight line through the origin in the direction of an eigenvector, then any vector on this straight line will remain on the line after the map A is applied.

<u>Example</u>: Find all eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$.

Solution: $\underbrace{Ax = \lambda x}$

$$\Rightarrow Ax - \lambda x = 0$$

$$\Rightarrow Ax - \lambda \cdot \underbrace{I_2}_{\text{identity matrix}} x = 0$$

(as $I_2 x = x$)

$$\Rightarrow (A - \lambda I_2)\, x = 0$$

This is a homogeneous system.
In order to obtain a nontrivial solution ($x \neq 0$), $A - \lambda I_2$ must be singular:

$$\det(A - \lambda I_2) = 0.$$

(because if $\det(\dots) \neq 0$, then we have exactly one solution, namely $x = 0$).

$$\Rightarrow A - \lambda I_2 = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} - \lambda \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{pmatrix}$$

$$\Rightarrow \det(A - \lambda I_2) = (1-\lambda)(2-\lambda) - 2 \cdot 3$$
$$= 2 - \lambda - 2\lambda + \lambda^2 - 6$$
$$= \lambda^2 - 3\lambda - 4$$
$$= (\lambda+1)(\lambda-4) \overset{!}{=} 0$$

$\Rightarrow \lambda_1 = -1,\ \lambda_2 = 4$. These are the eigenvalues of A.

To compute the eigenvectors, we solve

To compute the eige

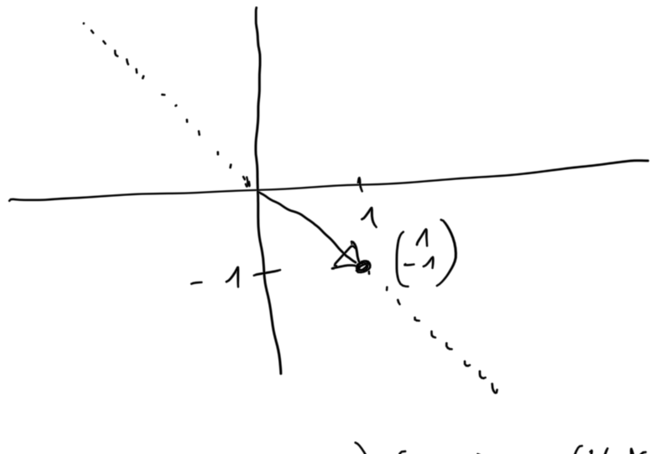
- $Ax = -x$ for $\lambda_1 = -1$
- $Ax = 4x$ for $\lambda_2 = 4$.

- $Ax = -x \implies \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}$

$$\implies \begin{aligned} x_1 + 2x_2 &= -x_1 \\ 3x_1 + 2x_2 &= -x_2 \end{aligned}$$

$$\implies \underbrace{\begin{aligned} 2x_1 + 2x_2 &= 0 \quad | :2 \\ 3x_1 + 3x_2 &= 0 \quad | :3 \end{aligned}}_{\boxed{x_1 + x_2 = 0}} \Bigg\} \text{ these equations are identical!}$$

$\implies$ All vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ that satisfy $x_1 + x_2 = 0$ are eigenvectors to the eigenvalue $\lambda_1 = -1$. For example, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector associated to the eigenvalue $\lambda_1 = -1$ of A.



- $Ax = 4x \implies \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4x_1 \\ 4x_2 \end{pmatrix}$

$$\implies \begin{aligned} x_1 + 2x_2 &= 4x_1 \\ 3x_1 + 2x_2 &= 4x_2 \end{aligned}$$

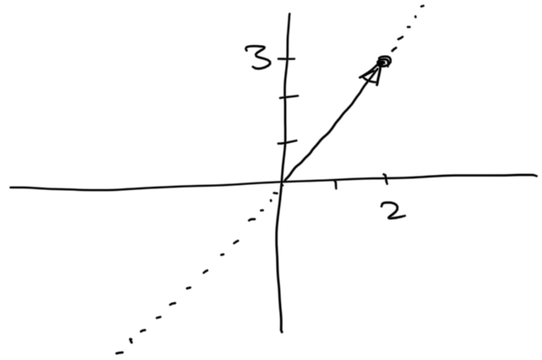
$$\implies -3x_1 + 2x_2 = 0 \quad | \cdot (-1) \Big\} \text{ these are}$$

$$3x_1 - 2x_2 = 0 \qquad \} \text{ identical}$$

$$\Rightarrow \boxed{3x_1 - 2x_2 = 0}$$

$\Rightarrow$ All vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ that satisfy $3x_1 = 2x_2$ are eigenvectors associated to the eigenvalue $\lambda_2 = 4$ of A.

For example, $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is an eigenvector.



Eigenvectors are not unique, they are determined only up to a multiplicative constant.

When the eigenvalues are real (as they are in this course), all eigenvectors corresponding to a particular eigenvalue lie on the same straight line through the origin.

Indeed, when x is an eigenvector of A corr. to the eigenvalue λ, then $\alpha \cdot x$ is also an eigenvector of A, $(\alpha \in \mathbb{R})$

because:

$$A \cdot (\underline{\alpha x}) = \alpha \cdot (\underline{Ax}) = \alpha \cdot (\underline{\lambda x})$$
$$= \lambda \cdot (\underline{\alpha x}).$$

<u>Example</u>: Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}$.

<u>Solution</u>: To find the eigenvalues of A, we solve

$$\det(A - \lambda I_2) = \det\begin{pmatrix} 1-\lambda & 4 \\ 1 & -2-\lambda \end{pmatrix}$$

$$= (1-\lambda)(-2-\lambda) - 4\cdot 1$$

$$= -2 - \lambda + 2\lambda + \lambda^2 - 4$$

$$= \lambda^2 + \lambda - 6$$

$$= (\lambda-2)(\lambda+3) \overset{!}{=} 0$$

$\Rightarrow \lambda_1 = 2,\ \lambda_2 = -3$ are the eigenvalues of A.

- eigenvectors corr. to $\lambda_1 = 2$:

$$Ax = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} x_1 + 4x_2 &= 2x_1 \\ x_1 - 2x_2 &= 2x_2 \end{aligned}\right\} \Rightarrow \underbrace{\begin{aligned} -x_1 + 4x_2 &= 0 \\ x_1 - 4x_2 &= 0 \end{aligned}}_{\text{both equations are identical}}$$

$$\Rightarrow \boxed{x_1 - 4x_2 = 0}$$

$\Rightarrow$ All vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with $x_1 = 4x_2$ are eigenvectors w.r.t. $\lambda_1 = 2$. For example, $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ is an eigenvector w.r.t. λ_1.

- eigenvectors corr. to $\lambda_2 = -3$:

$$Ax = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3x_1 \\ -3x_2 \end{pmatrix}$$

$\Rightarrow \quad x_1 + 4x_2 = -3x_1$

$\Rightarrow \quad 4x_1 + 4x_2 = 0 \quad \Rightarrow \quad \boxed{x_1 + x_2 = 0}$

$\Rightarrow$ All vectors $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with $x_1 = -x_2$ are eigenvectors w.r.t. $\lambda_2 = -3$.
For example, $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector w.r.t. λ_2.

When a matrix A is

- diagonal $\left(A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \right)$
- upper triangular $\left(A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right)$
- lower triangular $\left(A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \right)$,

then the eigenvalues are precisely the diagonal elements.

<u>Example</u>: Find the eigenvalues of

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

$$\Rightarrow \det(A - \lambda I_2) = \det\left(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} - \lambda \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= \det \begin{pmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{pmatrix}$$

$$= (1-\lambda)(2-\lambda) \overset{!}{=} 0$$

$\Rightarrow \lambda_1 = 1, \quad \lambda_2 = 2$

$\Rightarrow \lambda_1 = \dots$

<u>Example</u>: Find all eigenvalues and eigenvectors of $A = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}$.

- <u>Eigenvalues</u>:

$$\det(A - \lambda I_2) = \det\left(\begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)$$

$$= \det\begin{pmatrix} -2-\lambda & 1 \\ 0 & -1-\lambda \end{pmatrix}$$

$$= (-2-\lambda)(-1-\lambda) - 1 \cdot 0 \overset{!}{=} 0$$

$$\Rightarrow \lambda_1 = -2, \quad \lambda_2 = -1.$$

- <u>Eigenvectors corr. to $\lambda_1 = -2$</u>:

$$Ax = \lambda_1 x$$

$$\Rightarrow \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_1 \\ -2x_2 \end{pmatrix}$$

$$\Rightarrow \begin{array}{l} -2x_1 + \overset{=0}{\widetilde{x_2}} = -2x_1 \quad \Rightarrow -2x_1 = -2x \\ \quad\quad - x_2 = -2x_2 \quad \Rightarrow x_2 = 0 \end{array}$$

$\Rightarrow -2x_1 = -2x_1$: Any value of x_1 satisfies this equation!
However, we cannot choose $x_1 = 0$, as $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ cannot be an eigenvector.

$\Rightarrow$ E.g., $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector associated to $\lambda_1 = -2$.

- <u>Eigenvectors corr. to $\lambda_2 = -1$</u>:

$$A x = \lambda_2 x$$

$$\Rightarrow \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}$$

$$\Rightarrow \begin{array}{rl} -2x_1 + x_2 = -x_1 & \Rightarrow x_1 = x_2 \\ -x_2 = -x_2 & \end{array}$$

The second equation ($-x_2 = -x_2$) means that any value of x_2 is allowed. From the first equation, we get $x_1 = x_2$. This means that any vector $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ with $x_1 = x_2$, $x \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, is an eigenvector of A w.r.t. $\lambda_2 = -1$.

$\Rightarrow$ E.g., the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector.

Suppose that A is a square matrix and that x is an eigenvector of A corresponding to the eigenvalue λ:

$$A x = \lambda x$$

$$A^2 x = A(A x) = \lambda (A x) = \lambda^2 x$$

$$A^3 x = A(A^2 x) = A(\lambda^2 x) = \lambda^2 (A x) = \lambda^3 x$$

$$\vdots$$

$$\boxed{A^n x = \lambda^n x}$$ for any positive integer $n \geq 1$.

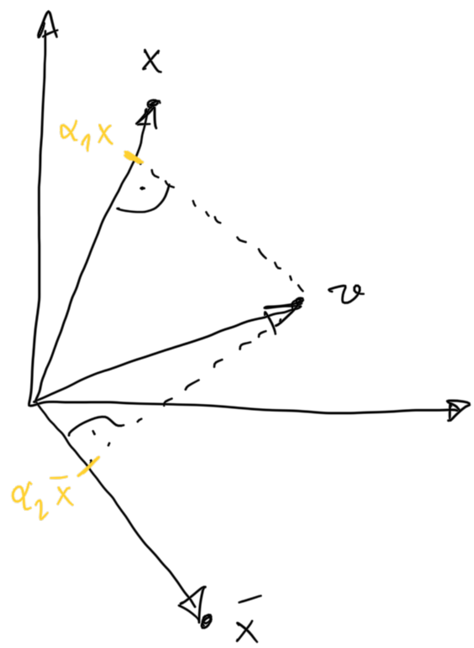
In other words: If x is an eigenvector

of A corr. to the eigenvalue λ,
x will also be an eigenvector of A^n corr. to the eigenvalue λ^n.

This analysis is useful if we want to find vectors of the form $A^N v$, where v is some arbitrary vector (not necessarily an eigenvector of A), and N is a big integer.

Suppose we are given some vector v. If we know two eigenvectors $x, \bar{x}$ of A, we can analyze v as

$$v = \alpha_1 \cdot x + \alpha_2 \cdot \bar{x}.$$



Then, with e.g. $N=1000$

$$A^{1000} v = A^{1000}(\alpha_1 x + \alpha_2 \bar{x})$$
$$= \alpha_1 (A^{1000} x) + \alpha_2 (A^{1000} \bar{x})$$
$$= \alpha_1 \lambda_1^{1000} x + \alpha_2 \lambda_2^{1000} \bar{x}$$

(it is much easier to calculate λ_1^{1000}, λ_2^{1000} than A^{1000}).

<u>Example</u>: Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$. Find $A^{10} v$, where $v = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

The eigenvalues of A are $\lambda_1 = -1$, $\lambda_2 = 4$

$\Rightarrow$ The eigenvalues of A

An eigenvector corr. to λ_1 is $\begin{pmatrix}1\\-1\end{pmatrix} =: x$

——— " ——— λ_2 is $\begin{pmatrix}2\\3\end{pmatrix} =: \overline{x}$

$\Rightarrow$ We write v as the linear combination of the vectors $\begin{pmatrix}1\\-1\end{pmatrix}$ and $\begin{pmatrix}2\\3\end{pmatrix}$:

$$v = \begin{pmatrix}4\\1\end{pmatrix} = \alpha_1 \cdot \begin{pmatrix}1\\-1\end{pmatrix} + \alpha_2 \cdot \begin{pmatrix}2\\3\end{pmatrix}$$

$$\Rightarrow \begin{array}{l} 4 = \alpha_1 + 2\alpha_2 \\ 1 = -\alpha_1 + 3\alpha_2 \end{array} \quad \Rightarrow \boxed{\alpha_1 = 4 - 2\alpha_2}$$

$$\Rightarrow 1 = -(4 - 2\alpha_2) + 3\alpha_2$$

$$\Rightarrow 1 = -4 + 5\alpha_2$$

$$\Rightarrow 5\alpha_2 = 5$$

$$\Rightarrow \underline{\underline{\alpha_2 = 1}}$$

$$\Rightarrow \underline{\underline{\alpha_1}} = 4 - 2 \cdot 1 = \underline{\underline{2}}$$

Therefore, $v = 2 \cdot \begin{pmatrix}1\\-1\end{pmatrix} + \begin{pmatrix}2\\3\end{pmatrix} = 2x + \overline{x}$.

We now find $A^{10} v$:

$$\underline{\underline{A^{10} v}} = A^{10}(2x + \overline{x})$$

$$= 2 \cdot A^{10} x + A^{10} \overline{x}$$

$$= 2 \cdot \lambda_1^{10} x + \lambda_2^{10} \overline{x}$$

$$= 2 \cdot (-1)^{10} x + 4^{10} \overline{x}$$

$$= 2 \cdot x + 4^{10} \overline{x}$$

$$= 2 \cdot \begin{pmatrix}1\\-1\end{pmatrix} + 4^{10} \cdot \begin{pmatrix}2\\3\end{pmatrix}$$

$$= \begin{pmatrix} 2 + 4^{10} \cdot 2 \\ -2 + 4^{10} \cdot 3 \end{pmatrix}$$