Linear Algebra

An Application of Matrices - Leslie Models

Elisabeth Köbis, elisabeth.kobis@ntnu.no

Population Models

So far, we have studied continuous-time population models: $N(t)$ was the size of the population, where $t \geq 0$ was the time.

There also exist **discrete time** models, for example, where $N(t)$ is the size of the population at time $t = 0, 1, 2, \ldots$:

$$\begin{aligned}
N(0) &= N_0 \\
N(1) &= R \cdot N(0) \\
N(2) &= R \cdot N(1) \\
N(3) &= R \cdot N(2) \\
&\vdots \\
N(t+1) &= R \cdot N(t),
\end{aligned}$$

where N_0 is the size of the population at time $t = 0$. The solution is $N(t) = R^t \cdot N_0$ ($t = 0, 1, 2, \ldots$).

Population Models

In the previous example, $N(t+1) = R \cdot N(t)$ suggests that on average, every individual will produce R new individuals per time unit. This does not take into account that the reproduction rate will be different for different ages. However, we often need to take into account the fact that the reproduction depends very highly on the age of the individuals.

Leslie models are discrete-time, age structured models.

Leslie Models - An Example

We study a species that does not live beyond age 3. Since only females produce offspring, we study only the number of females. Let

$N_i(t)$: number of age-i females at time $t = 0, 1, 2, \ldots$ $(i = 0, 1, 2, 3)$.

We do not study a number $N(t)$, but a vector

$$\mathbf{N}(t) = \begin{pmatrix} N_0(t) \\ N_1(t) \\ N_2(t) \\ N_3(t) \end{pmatrix}.$$

Leslie Models - An Example

For the population we study, we have the following observations:

Every year (or every reproductive session)

- 40% of the age-0 females survive,
- 30% of the age-1 females survive,
- 10% of the age-2 females survive.

So,

$$N_1(t+1) = 0.4N_0(t)$$
$$N_2(t+1) = 0.3N_1(t)$$
$$N_3(t+1) = 0.1N_2(t).$$

Females reproduce after the age of 1. The age-1 females give 2 females each on average. The age-2 females give 1.5 females each on average. This means

$$N_0(t+1) = 2N_1(t) + 1.5N_2(t).$$

Leslie Models - An Example

We can describe this via a matrix equation:

$$\begin{pmatrix} N_0(t+1) \\ N_1(t+1) \\ N_2(t+1) \\ N_3(t+1) \end{pmatrix} = \begin{pmatrix} 0 & 2 & 1.5 & 0 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{pmatrix} \begin{pmatrix} N_0(t) \\ N_1(t) \\ N_2(t) \\ N_3(t) \end{pmatrix},$$

or, equivalently,

$$\mathbf{N}(t+1) = L\mathbf{N}(t),$$

where

$$L = \begin{pmatrix} 0 & 2 & 1.5 & 0 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{pmatrix}$$

is the **Leslie matrix** of the model.

Leslie Models - An Example

For example, if at some time t in the previous model there is the following distribution of the population into the following age groups:

$$N_0(t) = 1000,\ N_1(t) = 200, N_2(t) = 100,\ N_3(t) = 10,$$

then the population at time $t+1$ will be

$$\begin{pmatrix} 0 & 2 & 1.5 & 0 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0.1 & 0 \end{pmatrix} \begin{pmatrix} 1000 \\ 200 \\ 100 \\ 10 \end{pmatrix} = \begin{pmatrix} 550 \\ 400 \\ 60 \\ 10 \end{pmatrix}.$$

Leslie Models - An Example

Conversely, if we are given the Leslie matrix of the model, we can deduce information of the population.
Suppose we are given the following Leslie matrix:

$$\begin{pmatrix} 5 & 7 & 1.5 \\ 0.2 & 0 & 0 \\ 0 & 0.4 & 0 \end{pmatrix}.$$

We interpret this as follows:

- the population is divided into 3 age groups (i.e., the females live for at most 2 years)
- 20% of the 0-year old females survive, 40% of the 1-year old females survive
- all age groups contribute to the reproduction of the species.
- The 0-year old females give 5 new females each on average
- The 1-year old females give 7 new females each on average
- The 2-year old females give 1.5 new females each on average.

Leslie Models - An Example

Question: Suppose a population described by the previous Leslie matrix starts with 1000 zero-aged females. What will the population consists of in 2 years?

Answer:

$$\mathbf{N}(0) = \begin{pmatrix} 1000 \\ 0 \\ 0 \end{pmatrix}.$$

$$\mathbf{N}(1) = \begin{pmatrix} 5 & 7 & 1.5 \\ 0.2 & 0 & 0 \\ 0 & 0.4 & 0 \end{pmatrix} \begin{pmatrix} 1000 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 5000 \\ 200 \\ 0 \end{pmatrix}.$$

$$\mathbf{N}(2) = \begin{pmatrix} 5 & 7 & 1.5 \\ 0.2 & 0 & 0 \\ 0 & 0.4 & 0 \end{pmatrix} \begin{pmatrix} 5000 \\ 200 \\ 0 \end{pmatrix} = \begin{pmatrix} 26400 \\ 1000 \\ 80 \end{pmatrix}.$$