Linear Algebra

Vectors. Linear Maps

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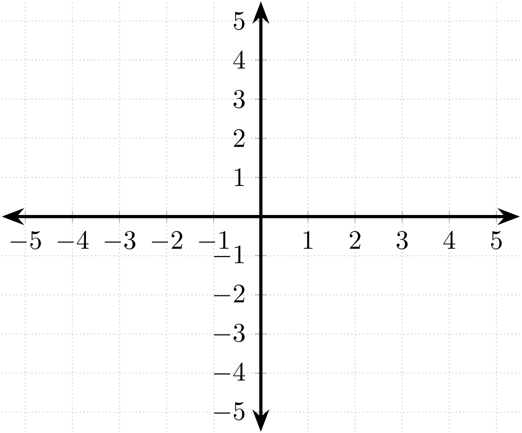
Vectors

Recall that a 2×1-matrix is also called a **column vector**. We will also call it 2-dimensional vector. Such a vector represents a point in the plane.

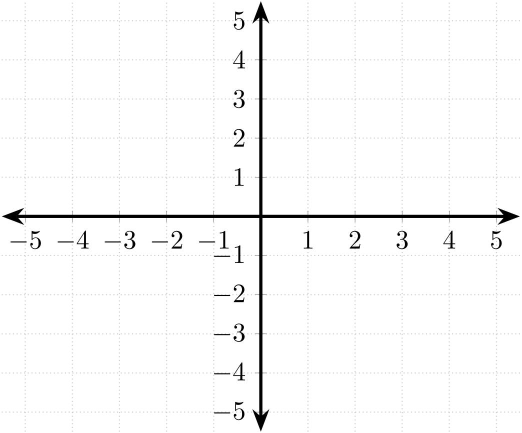
E.g., the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ represents the following point in the plane:



Vectors

There is a one-to-one correspondence between a vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and its point in the plane.

A vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ represents the following point in the plane and vice versa:



Vectors

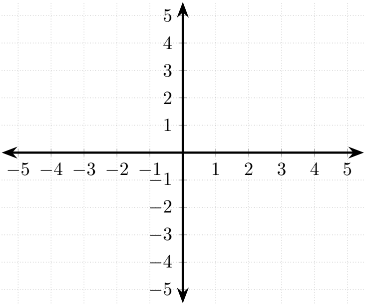
We already know addition of vectors:

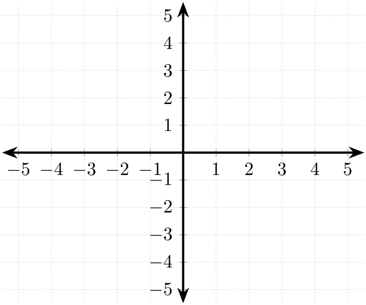
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$$

and multiplication of a scalar and a vector:

$$\lambda \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}.$$

We can view these operations in the plane as follows:





Vectors

When we are given two vectors x and y, the sum $x + y$ is the diagonal of the parallelogram with sides x and y.

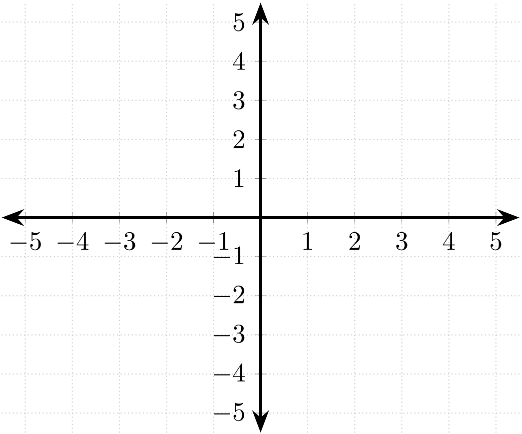
The operations *addition of vectors* and *multiplication of a vector by a scalar* are called vector operations or linear operations.

Example

Let

$$u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} -1 \\ -3 \end{pmatrix}, \quad w = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.$$

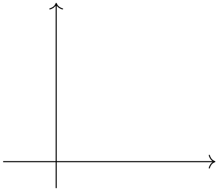
Find the vectors $-\frac{1}{2}u$, $u + v$, $v + w$ and $-v$ and illustrate them.



Vectors

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be a vector. The length of the vector x is

$$|x| = \sqrt{{x_1}^2 + {x_2}^2}.$$

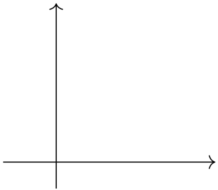


If λ is a scalar, then the length of the vector λx is

$$\begin{aligned} |\lambda x| &= \sqrt{(\lambda x_1)^2 + (\lambda x_2)^2} \\ &= \sqrt{\lambda^2({x_1}^2 + {x_2}^2)} \\ &= \sqrt{\lambda^2}\sqrt{({x_1}^2 + {x_2}^2)} \\ &= |\lambda|\sqrt{({x_1}^2 + {x_2}^2)} \\ &= |\lambda| \cdot |x|. \end{aligned}$$

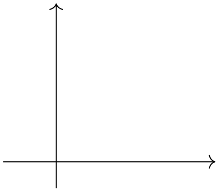
Vectors

We have seen that every point M in the plane corresponds to a unique pair (x_1, x_2) of real numbers. The pair (x_1, x_2) is called the **Cartesian coordinates** of the point M.



Vectors

Apart from the Cartesian coordinates, we can also use polar coordinates. Any point $M(x_1, x_2)$ has a distance $r \geq 0$ from the origin $(0, 0)$ and forms an angle θ with $0 \leq \theta \leq 2\pi$ with the positive horizontal axis (measured counterclockwise).



The numbers (r, θ) are uniquely determined and are given by the relations:

$$\boxed{r = \sqrt{{x_1}^2 + {x_2}^2}} \quad \text{and} \quad \boxed{\tan(\theta) = \frac{x_2}{x_1},\ 0 \leq \theta \leq 2\pi.}$$

The pair (r, θ) is called the **polar coordinates** of the point M.

Conversely, when we are given a point M with distance $r \geq 0$ from the origin that forms an angle $0 \leq \theta \leq 2\pi$ with the positive horizontal axis (measured counterclockwise), we can find the Cartesian coordinates of M by

$$x_1 = r\cos\theta,$$
$$x_2 = r\sin\theta.$$

Vectors

Example

The length of the vector $M = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is equal to 4 and the angle it forms with the positive horizontal axis (clockwise) is equal to $\frac{2\pi}{3}$. Find the Cartesian coordinates (x_1, x_2) of the vector M.

Solution: The polar coordinates of the point (x_1, x_2) are

$$r = 4$$
$$\theta = 2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}.$$

The Cartesian coordinates are

$$x_1 = r\cos\theta = 4\cos\left(\frac{4\pi}{3}\right) = -2$$
$$x_2 = r\sin\theta = 4\sin\left(\frac{4\pi}{3}\right) = -2\sqrt{3}.$$

Therefore $M = (-2, -2\sqrt{3})$.

Linear Maps

A function f of 2-dimensional vectors (meaning $f : \mathbb{R}^2 \to \mathbb{R}^2$) is called **linear** if it preserves the linear operations, that is, if for any 2-dimensional vectors $x, y \in \mathbb{R}^2$

1. $f(x + y) = f(x) + f(y)$;
2. $f(\lambda x) = \lambda f(x)$.

Example

Let A be a 2×2-matrix. Then the function

$$f(x) = Ax$$

is a linear function, because

1. $f(x + y) = A(x + y) = Ax + Ay$;
2. $f(\lambda x) = A(\lambda x) = \lambda Ax$.

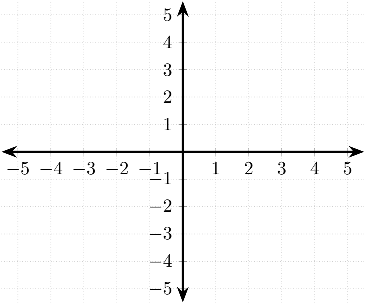
Therefore, a matrix can be viewed as a function that acts on vectors.

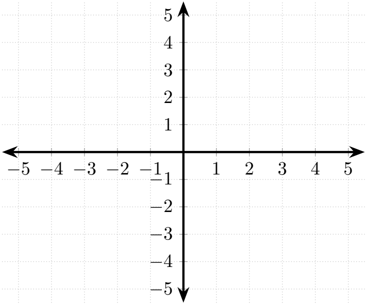
Linear Maps: The Identity Function

Let I_2 be the 2×2-identity matrix. Then the identity function is

$$f(x) = I_2 x = x$$

for any 2-dimensional vector x.



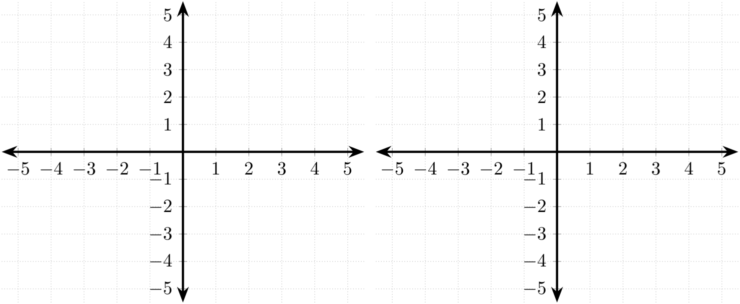


Linear Maps: Example

Consider the diagonal matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and let $f(x) = Ax$. Then

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 \\ bx_2 \end{pmatrix}.$$

So, f multiplies the first coordinate by a and the second by b. E.g., for $\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$, we have:



Example: Rotation Map

We now find the matrix R_θ that rotates every vector x by an angle θ (counterclockwise). Assume the coordinates of x are

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} r\cos\alpha \\ r\sin\alpha \end{pmatrix}.$$

The polar coordinates of the vector $R_\theta x$ are $(r, \alpha+\theta)$, hence, its Cartesian coordinates are

$$\begin{aligned} \begin{pmatrix} r\cos(\alpha+\theta) \\ r\sin(\alpha+\theta) \end{pmatrix} &= \begin{pmatrix} r\cos\alpha\cos\theta - r\sin\alpha\sin\theta \\ r\sin\alpha\cos\theta + r\cos\alpha\sin\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta\cdot x_1 - \sin\theta\cdot x_2 \\ \cos\theta\cdot x_2 + \sin\theta\cdot x_1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

So, $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

Linear Maps

Question: Is every map that transfers one vector into another vector a *linear* map?

Answer: No. Take, for example, the map

$$f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} {x_1}^2 \\ {x_2}^2 \end{pmatrix}.$$

Then we get for $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ that

$$f(x) = x, \text{ but } f(-x) = x.$$

If f were a linear function, we should have $f(-x) = -f(x)$, which is not true here.