Lecture 12

Determinants

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix.
Then the <u>determinant</u> of the matrix A is defined as the number

$$\det A := |A| := ad - bc.$$

<u>Examples</u>: • Let $A = \begin{pmatrix} -3 & 1 \\ 1 & 1 \end{pmatrix}$. Then

$$\det A = -3\cdot 1 - 1\cdot 1 = \underline{\underline{-4}}$$

• Let $B = \begin{pmatrix} 0 & -1 \\ 5 & 4 \end{pmatrix}$. Then

$$\det B = 0\cdot 4 - (-1)\cdot 5 = \underline{\underline{5}}$$

<u>Theorem</u>: The 2×2 matrix A is invertible if and only if $\det A \neq 0$.

<u>Examples</u>: Which of the following matrices are invertible, and which ones are singular?

• $A = \begin{pmatrix} 5 & 2 \\ -1 & 6 \end{pmatrix} \Rightarrow \det A = 5\cdot 6 - 2\cdot(-1) = 32 \neq 0$

$\Rightarrow A$ is invertible.

$\det B = 3\cdot(-6) - (-2)\cdot 9$

- $B = \begin{pmatrix} 3 & -2 \\ 9 & -6 \end{pmatrix} \Rightarrow \det B =$

$= 0$

$\Rightarrow B$ is singular.

- $C = \begin{pmatrix} 5 & 7 \\ -1 & 3/2 \end{pmatrix} \Rightarrow \det C = 5 \cdot 3/2 - 7 \cdot (-1)$

$= 14\,1/2 \neq 0$

$\Rightarrow C$ is invertible.

- $D = \begin{pmatrix} 101 & 0 \\ -51 & 2 \end{pmatrix} = 101 \cdot 2 - 0 \cdot (-51)$

$= 202 \neq 0$

$\Rightarrow D$ is invertible.

<u>Theorem</u>: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If A is invertible (i.e., if $ad - bc \neq 0$), then the inverse of A is

$$\boxed{A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}.$$

<u>Examples</u>: (i) Find the inverse of $A = \begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$.

$\det A = 3 \cdot 4 - 5 \cdot 2 = 12 - 10 = 2 \neq 0$

$$A^{-1} = \frac{1}{2} \begin{pmatrix} 4 & -5 \\ -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -5/2 \\ -1 & 3/2 \end{pmatrix}$$

(ii) Find the inverse of $B = \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$.

$\det B = 2 \cdot 3 - 6 \cdot 1 = 0$

$\Rightarrow B$ is not invertible.

<u>Example</u>: For which value of $t \in \mathbb{R}$ is the matrix

$$A_t = \begin{pmatrix} t & -2 \\ 3 & t+5 \end{pmatrix}$$

invertible? For these values of t, find A_t^{-1}.

- We find the determinant of A_t:

$$\underline{\underline{\det A_t}} = t \cdot (t+5) - (-2) \cdot 3$$
$$= t^2 + 5t + 6$$
$$= \underline{\underline{(t+2)(t+3)}}.$$

The matrix A_t is invertible <u>if and only if</u>

$$\det A_t \neq 0 \iff (t+2)(t+3) \neq 0$$
$$\iff t \neq -2 \text{ and } t \neq -3.$$

So, A_t is invertible whenever $t \neq -2$, $t \neq -3$.

For value $t \neq -2$, $t \neq -3$, we have

$$A_t^{-1} = \frac{1}{(t+2)(t+3)} \cdot \begin{pmatrix} t+5 & 2 \\ -3 & t \end{pmatrix}.$$

Recall that a 2×2 linear system

$$a_{11}x + a_{12}y = b_1$$
$$a_{21}x + a_{22}y = b_2$$

can also be written in matrix form as

$$AX = b$$

with $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

When A is invertible, we can multiply both sides of the system by A^{-1}:

$$AX = b \implies A^{-1}(AX) = A^{-1}b$$
$$\implies \underbrace{(A^{-1}A)}_{I_2}X = A^{-1}b$$
$$\implies I_2 X = A^{-1}b$$
$$\implies \underline{\underline{X = A^{-1}b}}$$

The system has the unique solution $X = A^{-1}b$.

- When A is not invertible, the system $AX = b$ will either have no solutions or infinitely many solutions.

<u>Example</u>: Write the system

$$2x - y = 1$$
$$x + 2y = -2$$

in matrix form. Solve it using matrices.
$\implies$ This system is equivalent to $AX = b$ with

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

We find the inverse of A:

$\det A = 2 \cdot 2 - (-1) \cdot 1 = 5 \neq 0$,

so A is invertible. We get

$$A^{-1} = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}.$$

We solve the system:

$$\begin{aligned} AX = b \implies X &= A^{-1} b \\ &= \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 2 \cdot 1 - 2 \cdot 1 \\ -1 \cdot 1 - 2 \cdot 2 \end{pmatrix} \\ &= \frac{1}{5} \cdot \begin{pmatrix} 0 \\ -5 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 \\ -1 \end{pmatrix}}} \end{aligned}$$

$\implies (x, y) = (0, -1)$ solves the system.

- When $b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the 2×2 system $AX = b$ is called a <u>homogeneous</u> 2×2 system.

In other words, a homogeneous 2×2 system has the form

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0. \end{aligned}$$

(with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$).

A homogeneous system always has the trivial solution $(x, y) = (0, 0)$.

- When $\det A \neq 0$, the solution $(x, y) = (0,0)$ is unique.
- When $\det A = 0$, the system has infinitely many solutions.

<u>Example</u>: Let $A = \begin{pmatrix} 2 & 4 \\ 1 & a \end{pmatrix}$. Find the value of $a \in \mathbb{R}$ such that the system $\underline{AX = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$ has a unique solution.

$\Rightarrow$ The system $AX = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a homogeneous system. It has a unique solution precisely when A is invertible.

$$\det A = 2 \cdot a - 4 \cdot 1 = 2a - 4$$
$$= 2(a-2).$$

$\det A = 0 \iff a = 2$.

So, the system has a unique solution if $a \neq 2$.

If $a = 2$, $\det A = 0$ and the homogeneous system has infinitely many solutions.

<u>3x3 Square Matrices</u>

Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. Then we define

$$\det A := |A| := \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$= a \cdot \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \cdot \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \cdot \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg)$$

<u>Example</u>: $B = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & 1 \\ -1 & -1 & 1 \end{pmatrix}$

$$\rightarrow \det B = \underline{1 \cdot \begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix}} - 1 \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 0 & 3 \\ -1 & -1 \end{vmatrix}$$

$$= 4 - 1 - 3 = 0.$$

As for 2×2 matrices, A is invertible if and only if $\det A \neq 0$.

<u>Example</u>: Let $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix}$.

$$\det A = 1 \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} - 0 \cdot \begin{vmatrix} \dots \end{vmatrix} + 1 \cdot \begin{vmatrix} -1 & 1 \\ 1 & 3 \end{vmatrix}$$

$$= -4 - 4 = -8 \neq 0$$

$\rightarrow A$ is invertible.

Finding the inverse of A:

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}.$$

$$\left(\begin{array}{ccc|ccc} a & b & c & 1 & 0 & 0 \\ d & e & f & 0 & 1 & 0 \end{array}\right) \sim \dots$$ (Gauss elimination ...)

$$\left(\begin{array}{ccc:ccc} \multicolumn{3}{c}{\underbrace{g \quad h \quad i}_{A}} & \multicolumn{3}{c}{\underbrace{0 \quad 0 \quad 1}_{I_3}} \end{array}\right)$$

process)

$$\cdots \sim \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & - & - & - \\ 0 & 1 & 0 & - & - & - \\ 0 & 0 & 1 & \multicolumn{3}{c}{\underbrace{- \quad - \quad -}_{A^{-1}}} \end{array}\right)$$

<u>Example</u>: $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix}$.

$$\left(\begin{array}{ccc:ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right) \underset{R3 \to R3 - R1}{\overset{R2 \to R2 + R1}{\sim}} \left(\begin{array}{ccc:ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 3 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\overset{R3 \to R3 - 3 \cdot R2}{\sim} \left(\begin{array}{ccc:ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & -8 & -4 & -3 & 1 \end{array}\right)$$

$$\overset{R3 \to \left(-\frac{1}{8}\right) \cdot R3}{\sim} \left(\begin{array}{ccc:ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{3}{8} & -\frac{1}{8} \end{array}\right)$$

$$\underset{\sim}{\overset{R1 \to R1 - R3}{R2 \to R2 - 3R3}} \left(\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{1}{2} & -3/8 & \frac{1}{8} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{8} & 3/8 \\ 0 & 0 & 1 & \frac{1}{2} & \frac{3}{8} & -\frac{1}{8} \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 0 & 0 & 1 & \frac{1}{2} & 3/8 & -1/8 \end{array}\right)$$
$$\underbrace{\qquad\qquad}_{=A^{-1}}$$

So, $A^{-1} = \begin{pmatrix} 1/2 & -3/8 & 1/8 \\ -1/2 & -1/8 & 3/8 \\ 1/2 & 3/8 & -1/8 \end{pmatrix}$

$$= \frac{1}{8}\begin{pmatrix} 4 & -3 & 1 \\ -4 & -1 & 3 \\ 4 & 3 & -1 \end{pmatrix}.$$

Let us check if $A \cdot A^{-1} = I_3 = A^{-1} \cdot A$

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 2 \\ 1 & 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4 & -3 & 1 \\ -4 & -1 & 3 \\ 4 & 3 & -1 \end{pmatrix} \cdot \frac{1}{8}$$

$$= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{pmatrix} \cdot \frac{1}{8} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

If we try to apply this method to find the inverse of a non-invertible (singular) matrix, then this method will stop because some row will be equal to zero.

E.g. $B = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 4 \end{pmatrix}$.

Then $\det B = 1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix} - 0 \cdot |\ldots| + 2 \cdot \begin{vmatrix} 1 \\ 2 \end{vmatrix}$

$$= 4 - 4 = 0.$$

So, B is not invertible.

... apply the previous method

If we try to apply the previous method, we get:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 4 & 0 & 0 & 1 \end{array}\right) \overset{R3 \to \frac{1}{2}\cdot R3}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & \frac{1}{2} \end{array}\right)$$

$$\overset{R3 \to R3 - R1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ \boxed{0 \quad 0 \quad 0} & & & -1 & 0 & 1/2 \end{array}\right)$$

We cannot continue from that point, because the third row has all entries equal to zero.

<u>Example</u>: Consider the system

$$\begin{aligned} 2x - 3y + z &= -1 \\ x + y - 2z &= -3 \\ 3x - 2y + z &= 2. \end{aligned}$$

(i) Write this system in the form $AX = b$.

(ii) Find the determinant of A.

(iii) Solve the system using the inverse of A.

(i) $A = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 1 & -2 \\ 3 & -2 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $b = \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}$.

(ii) $$\begin{vmatrix} 2 & -3 & 1 \\ 1 & 1 & -2 \\ 3 & -2 & 1 \end{vmatrix} = 2 \cdot \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} - (-3) \cdot \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}$$

$$2 \cdot (-7) + 3 \cdot 7 + 1 \cdot (-5)$$

$= $ [illegible]

$= \underline{\underline{10}}$

(iii) We find A^{-1}:

$$\left(\begin{array}{ccc|ccc} 2 & -3 & 1 & 1 & 0 & 0 \\ 1 & 1 & -2 & 0 & 1 & 0 \\ 3 & -2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$R1 \leftrightarrow R2$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 0 & 1 & 0 \\ 2 & -3 & 1 & 1 & 0 & 0 \\ 3 & -2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$R2 \to R2 - 2R1$
$R3 \to R3 - 3R1$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & -5 & 5 & 1 & -2 & 0 \\ 0 & -5 & 7 & 0 & -3 & 1 \end{array}\right)$$

$R3 \to R3 - R2$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & -5 & 5 & 1 & -2 & 0 \\ 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right)$$

$R3 \to \frac{1}{2} \cdot R3$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & -5 & 5 & 1 & -2 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right)$$

$R2 \to R2 - 5R3$

$R1 \to R1 + 2R3$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & -5 & 0 & 7/2 & 1/2 & -5/2 \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right)$$

$R2 \to \left(-\frac{1}{5}\right) \cdot R2$

$R1 - R2$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/10 & 1/10 & 1/2 \\ 0 & 1 & 0 & -7/10 & -1/10 & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right)$$

$$\underbrace{}_{= A^{-1}}$$

$$\underline{\underline{x}} = A^{-1} \cdot b = \underbrace{\frac{1}{10} \cdot \begin{pmatrix} -3 & 1 & 1 \\ -7 & -1 & 5 \\ -5 & -5 & 5 \end{pmatrix}}_{= A^{-1}} \cdot \begin{pmatrix} -1 \\ -3 \\ 2 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}}$$