Linear Algebra

Matrix Multiplication

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Multiplication of Matrices

First, let us remember the scalar product of two vectors. If $\overrightarrow{a} = (a_1, a_2)$, $\overrightarrow{b} = (b_1, b_2)$, then the **scalar product** of $\overrightarrow{a}$ and $\overrightarrow{b}$ (also called inner product or dual product) is

$$\overrightarrow{a} \cdot \overrightarrow{b} = a_1 b_1 + a_2 b_2.$$

E.g., if $\overrightarrow{a} = (2, 1)$, $\overrightarrow{b} = (-1, 3)$, then $\overrightarrow{a} \cdot \overrightarrow{b} = 2 \cdot (-1) + 1 \cdot 3 = 1$.

Multiplication of Matrices

The same definition of scalar product can be extended to n-dimensional vectors:

If $\overrightarrow{x} = (x_1, x_2, \ldots, x_n)$, $\overrightarrow{y} = (y_1, y_2, \ldots, y_n)$, then the **scalar product** of $\overrightarrow{x}$ and $\overrightarrow{y}$ is

$$\overrightarrow{x} \cdot \overrightarrow{y} = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n = \sum_{i=1}^{n} x_i y_i.$$

E.g., if $\overrightarrow{a} = (1, 0, 2, -1)$, $\overrightarrow{b} = (2, -3, 1, 0)$, then $\overrightarrow{a} \cdot \overrightarrow{b} = 1 \cdot 2 + 0 + 2 \cdot 1 + 0 = 4$.

Note that the scalar product is only defined for vectors of the same dimension.

Multiplication of Matrices

Suppose that A is an $m \times n$ matrix and B is an $n \times p$ matrix:

$$A = (a_{ij})_{i \leq m,\ j \leq n}, \quad B = (b_{ij})_{i \leq n,\ j \leq p}.$$

We define the product $A \cdot B$ to be the $m \times p$ matrix $A \cdot B = (c_{ij})_{i \leq m,\ j \leq p}$, where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

(i.e., c_{ij} is the scalar product of the i-th row of A with the j-th column of B).

Multiplication of Matrices

Example 1

Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix}.$$

Calculate $A \cdot B$.

A is a 2×2 matrix, B is a 2×3 matrix, so $A \cdot B$ will be a 2×3 matrix.

$$A \cdot B = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix} =$$

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Multiplication of Matrices

Example 2

Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix}.$$

What is $B \cdot A$?

B is 2×3, A is 2×2, so the product $B \cdot A$ is not defined. Remember: We can only multiply an $m \times n$ matrix with an $n \times p$ matrix, and the product will be an $m \times p$ matrix.

Multiplication of Matrices

Example 3

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 & -3 \\ 0 & -1 & 4 & 0 \\ -1 & 0 & -2 & 1 \end{pmatrix}.$$

What is $A \cdot B$?

A is 2×3, B is 3×4, so the product $A \cdot B$ will be a 2×4 matrix.

$$A \cdot B = \begin{pmatrix} -2 & 0 & 5 & 0 \\ -5 & -2 & -11 & 7 \end{pmatrix}.$$

Multiplication of Matrices

Example 4

Let

$$A = \begin{pmatrix} 2 & 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Find $A \cdot B$ and $B \cdot A$. Are these equal?

A is 1×3 (row vector), B is 3×1 (column vector), so the product $A \cdot B$ will be a 1×1 matrix (hence, a scalar) and the product $B \cdot A$ will be a 3×3 matrix. Hence, $A \cdot B$ and $B \cdot A$ cannot be equal.

$$A \cdot B = \begin{pmatrix} 2 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = (1).$$

$$B \cdot A = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \\ -2 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Multiplication of Matrices

Example 5

Let

$$A = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}.$$

Find $A \cdot B$ and $B \cdot A$. Is $A \cdot B = B \cdot A$?

$$A \cdot B = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0_{2\times 2}.$$

$$B \cdot A = \begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -4 & 2 \end{pmatrix}.$$

So, $AB \neq BA$.

Multiplication of Matrices

This last example shows the following:

1. When A, B are $n \times n$ (square) matrices, then AB and BA are not necessarily equal, i.e., matrix multiplication is not commutative (unlike multiplications of real numbers).
2. When a, b are real numbers and $a \cdot b = 0$, then $a = 0$ or $b = 0$. This does not happen with matrix multiplication: $A \cdot B = 0$ does not necessarily imply $A = 0$ or $B = 0$.

Multiplication of Matrices

The following properties hold whenever the products that are appearing are well-defined (i.e., if multiplication of the corresponding matrices is acceptable):

1. $(A + B) \cdot C = A \cdot C + B \cdot C$.
2. $A \cdot (B + C) = A \cdot B + A \cdot C$.
3. $A \cdot (B \cdot C) = (A \cdot B) \cdot C$.
4. $A \cdot 0 = 0 \cdot A = 0$.

Multiplication of Matrices

When A is a square $(n \times n)$ matrix, we can define the powers of A as follows:

$$\begin{aligned} A^2 &= A \cdot A \\ A^3 &= A^2 \cdot A \\ &\vdots \\ A^{n+1} &= A^n \cdot A. \end{aligned}$$

E.g., if $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then

$$A^2 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix},$$

$$A^3 = \begin{pmatrix} 7 & 10 \\ 15 & 22 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 37 & 54 \\ 81 & 118 \end{pmatrix}.$$

Identity Matrix

We define the $n \times n$ **identity matrix** to be

$$I_n = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Multiplication with the Identity Matrix

Whenever A is an $n \times n$ matrix, we have

$$A \cdot I_n = I_n \cdot A = A.$$

E.g., if $A = \begin{pmatrix} 1 & -10 \\ 2 & 21 \end{pmatrix}$, then

$$I_2 \cdot A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -10 \\ 2 & 21 \end{pmatrix} = \begin{pmatrix} 1 & -10 \\ 2 & 21 \end{pmatrix} = A$$

and

$$A \cdot I_2 = \begin{pmatrix} 1 & -10 \\ 2 & 21 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -10 \\ 2 & 21 \end{pmatrix} = A$$

I_n is called the identity matrix because it is the identity element of matrix multiplication (like 1 is for multiplication of numbers).

Multiplication with the Identity Matrix

Whenever M is an $m \times n$ matrix, we have

$$I_m \cdot M = M \quad \text{and} \quad M \cdot I_n = M.$$

Note that when $m \neq n$, I_m and I_n are matrices of different dimensions. For example,

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Linear Systems

Matrix multiplication can be used to write a linear system in a simpler form. Consider the system

$$\begin{aligned} a_{11}x_1 + a_{21}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m. \end{aligned}$$

If we set

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Then the above system can be written as

$$AX = b,$$

where we have to find X.

Linear Systems

Example

Write the system

$$\begin{aligned} 2x + 3y + 4z &= 1 \\ -x + 5y - 6z &= 7 \end{aligned}$$

in matrix form.

Solution: This system can be written as $AX = b$, where

$$A = \begin{pmatrix} 2 & 3 & 4 \\ -1 & 5 & -6 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 7 \end{pmatrix}.$$

Inverse Matrix

When we have an equation

$$ax = b$$

with $a, b \in \mathbb{R}$ and $a \neq 0$, then we can multiply the equation with a^{-1} (the inverse of a) and get the solution $x = \frac{b}{a}$. What about matrix equations of the form $AX = b$?

Let A be an $n \times n$ matrix. If there exists an $n \times n$ matrix B such that

$$AB = BA = I_n,$$

we say that B is the **inverse** matrix of A and we write $B = A^{-1}$:

$$A \cdot A^{-1} = A^{-1} \cdot A = I_n.$$

Inverse Matrix

Not all square matrices have an inverse. If A has an inverse, A is called an **invertible** matrix. If A does not have an inverse, A is called a **singular** matrix.

If the inverse of A exists, it is unique.

Example

Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. We have

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}}_{=A^{-1}} = \underbrace{\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}}_{=A^{-1}} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

Inverse Matrix

The following properties hold:

1. If A is invertible, then $(A^{-1})^{-1} = A$.
2. If A, B are both invertible $n \times n$ matrices, then $A \cdot B$ is also invertible and
$$(AB)^{-1} = B^{-1} \cdot A^{-1}.$$

Proof:

1. We have $A \cdot A^{-1} = I_n$, and so, A must be the inverse of A^{-1}, so $A = (A^{-1})^{-1}$.
2. $$\begin{aligned}(AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} \\ &= AI_nA^{-1} \\ &= AA^{-1} \\ &= I_n.\end{aligned}$$

Inverse Matrix

Example

Find the inverse of $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$.

Solution: We know that $AA^{-1} = I_2$.

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\implies \quad \begin{pmatrix} 2a+5c & 2b+5d \\ a+3c & b+3d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This means

$$2a+5c = 1, \quad 2b+5d = 0$$
$$a+3c = 0, \quad b+3d = 1,$$

giving $a = 3$, $c = -1$, $b = -5$ and $d = 2$. Thus,

$$A^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}.$$