Lecture 10

Examples for solving linear systems

(i) Solve
$$\begin{aligned} 2x + 2y - z &= 1 \\ 2x - y + z &= 2 \end{aligned}$$

$$\begin{aligned} 2x + 2y - z &= 1 \\ 2x - y + z &= 2 \end{aligned} \qquad \Big|\ R2 \to R1 - R2$$

$$\boxed{\begin{aligned} 2x + 2y - z &= 1 \\ 0 \quad 3y - 2z &= -1 \end{aligned}}$$

We set $z = t$, $t \in \mathbb{R}$.

- $3y - 2z = -1 \implies 3y = 2t - 1$
$$y = \frac{2}{3}t - \frac{1}{3} = \frac{2t-1}{3}$$

- $2x + 2y - z = 1$
$$\implies 2x + 2\left(\frac{2}{3}t - \frac{1}{3}\right) - t = 1$$
$$\implies 2x + \frac{4}{3}t - \frac{2}{3} - t = 1$$
$$\implies 2x + \frac{1}{3}t = \frac{5}{3}$$
$$\implies 2x = -\frac{1}{3}t + \frac{5}{3}$$
$$\implies x = -\frac{1}{6}t + \frac{5}{6} = \frac{5-t}{6}$$

$\Rightarrow$ The system has infinitely many solutions of the form

$$(x, y, z) = \left(\frac{5-t}{6}, \frac{2t-1}{3}, t \right), \quad t \in \mathbb{R}.$$

(ii) Solve
$$\begin{aligned} 2x - y &= 1 \\ x + y &= 2 \\ x - y &= 3 \end{aligned}$$

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 2 \\ x - y &= 3 \end{aligned} \quad \Bigg| \quad R3 \to R2 - R3$$

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 2 \\ 0 \quad 2y &= -1 \end{aligned} \Rightarrow \underline{\underline{y = -\frac{1}{2}}}$$

$$x - \frac{1}{2} = 2 \Rightarrow \underline{\underline{x = \frac{5}{2}}}$$

With these values of x and y, the first equation gives

$$\cancel{2} \cdot \frac{5}{\cancel{2}} - \left(-\frac{1}{2}\right) = 5.5 \neq 1,$$

i.e., the first equation does not hold true.

$\Rightarrow$ Hence, there do not exist x, y that satisfy all three equations at the same time.

$\Rightarrow$ The system has no solutions.

Matrices

We define an <u>$m \times n$ matrix</u> to be a rectangular array of numbers

$$A = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{n \text{ colums}} \left.\vphantom{\begin{pmatrix} a \\ a \\ \vdots \\ a \end{pmatrix}}\right\} m \text{ rows}$$

of m rows and n columns.

The real numbers a_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$, are called the entries of A.

The entry a_{ij} of the matrix A is located in the i-th row and the j-th column of A.

We also write $A = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$.

Examples: • $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & -1 & 5 \end{pmatrix}$ is a <u>2×3</u> matrix

(i.e., 2 rows, 3 columns).

E.g. $a_{21} = 0$

second row   first column

$a_{12} = 2$

• $B = \begin{pmatrix} 1 & 5 & 3 \\ -1 & -1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$ is a 3×3 matrix

(i.e., 3 rows, 3 columns)

E.g. $a_{31} = 2$.

An $n \times n$ matrix is called a <u>square</u> matrix (i.e., the number of rows is equal to the number of columns).

For example, $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, are both square matrices.

The <u>diagonal</u> of a square matrix consists of the elements of the form

$$a_{ii} \quad (i = 1, \dots, n).$$

E.g. $\begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

An $m \times 1$ matrix is called a column vector. $\begin{pmatrix} \vdots \end{pmatrix}$

A $1 \times n$ matrix is called a row vector. $\begin{pmatrix} \dots \end{pmatrix}$

E.g. $\begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix}$ is a column vector.

$\begin{pmatrix} 6 & 2 & -1 \end{pmatrix}$ is a row vector.

Using the Gauss elimination process to solve a linear system becomes much easier with the help of matrices.

Every linear system of the form

$$\boxed{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}}$$

$$\iff Ax = b \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

has an <u>augmented matrix</u>:

$$\left(\begin{array}{cccc:c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \underbrace{a_{m1}}_{} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

$$\underbrace{\hphantom{a_{m1}\ a_{m2}\ \dots\ a_{mn}}}_{A} \qquad \underbrace{\hphantom{b_m}}_{b}$$

Instead of applying the operations of the Gauss elimination method to the equations, we can apply them to the rows of the augmented matrix.
After applying the acceptable row operations, we arrive at an equivalent system.

<u>Example</u>:

$$\begin{array}{rcrcrcl} 3x & + & 5y & - & z & = & 10 \\ 2x & - & y & + & 3z & = & 9 \\ 4x & + & 2y & - & 3z & = & 1 \end{array}$$

Augmented Matrix:

$$\left(\begin{array}{ccc:c} 3 & 5 & -1 & 10 \end{array}\right)$$

$$\left(\begin{array}{ccc:c} 2 & -1 & 3 & 9 \\ 4 & 2 & -3 & 1 \end{array}\right)$$

$$R1 \to \tfrac{1}{3} R1 \quad \xrightarrow{\quad} \left(\begin{array}{ccc:c} 1 & 5/3 & -1/3 & 10/3 \\ 2 & -1 & 3 & 9 \\ 4 & 2 & -3 & 1 \end{array}\right)$$

$$\begin{array}{l} R2 \to R2 - 2 \cdot R1 \\ \xrightarrow{\qquad} \\ R3 \to R3 - 4 \cdot R1 \\ \xrightarrow{\qquad} \end{array} \left(\begin{array}{ccc:c} 1 & 5/3 & -1/3 & 10/3 \\ 0 & -13/3 & 11/3 & 7/3 \\ 0 & -14/3 & -5/3 & -43/3 \end{array}\right)$$

$$\begin{array}{l} R2 \to -\frac{3}{13} \cdot R2 \\ \xrightarrow{\qquad} \\ R3 \to -3 \cdot R3 \\ \xrightarrow{\qquad} \end{array} \left(\begin{array}{ccc:c} 1 & 5/3 & -1/3 & 10/3 \\ 0 & 1 & -11/13 & -7/13 \\ 0 & 14 & 5 & 43 \end{array}\right)$$

$$R3 \to R3 - 14 \cdot R2 \left(\begin{array}{ccc:c} 1 & 5/3 & -1/3 & 10/3 \\ 0 & 1 & -11/13 & -7/13 \\ 0 & 0 & \frac{219}{13} & \frac{657}{13} \end{array}\right)$$

$$R3 \to R3 \cdot \frac{13}{219} \left(\begin{array}{ccc:c} 1 & 5/3 & -1/3 & 10/3 \\ 0 & 1 & -11/13 & -7/13 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

The equivalent system is:

$$x + 5/3\, y - 1/3\, z = 10/3$$

$$y - \frac{11}{13} z = -\frac{7}{13}$$
$$z = 3$$

The solution is $(x, y, z) = (1, 2, 3)$.

Example:

$$\begin{aligned} 2x - 3y + z &= -1 \\ x + y - 2z &= -3 \\ 3x - 2y + z &= 2 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 2 & -3 & 1 & -1 \\ 1 & 1 & -2 & -3 \\ 3 & -2 & 1 & 2 \end{array}\right) \overset{R1 \leftrightarrow R2}{\sim} \left(\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 2 & -3 & 1 & -1 \\ 3 & -2 & 1 & 2 \end{array}\right)$$

$$\xrightarrow[R3 \to 3 \cdot R1 - R3]{R2 \to 2R1 - R2} \left(\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 0 & 5 & -5 & -5 \\ 0 & 5 & -7 & -11 \end{array}\right)$$

$$\begin{array}{l} R2 \to \frac{1}{5} \cdot R2 \\ R3 \to \frac{1}{5} \cdot R3 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -7/5 & -11/5 \end{array}\right)$$

$$R3 \to R2 - R3 \left(\begin{array}{ccc|c} 1 & 1 & -2 & -3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & \frac{2}{5} & 6/5 \end{array}\right)$$

$$\Rightarrow \quad x + y - 2z = -3 \Rightarrow \underline{\underline{x = 1}}$$
$$y - z = -1 \qquad \Rightarrow \underline{\underline{y = 2}}$$

$$\frac{2}{5}\cdot z = \frac{6}{5} \implies \underline{\underline{z = 3}}$$

Operations with Matrices

(I) Addition of Matrices

Suppose $A = (a_{ij})_{\substack{i\le m\\ j\le n}}$, $B = (b_{ij})_{\substack{i\le m\\ j\le n}}$

are two $\underline{m\times n}$ matrices.

We define the <u>sum</u> of A and B to be the $m\times n$ matrix

$$A+B = (a_{ij}+b_{ij})_{\substack{i\le m\\ j\le n}}.$$

E.g. If $A = \begin{pmatrix} \boxed{2} & 1 & -1 \\ 0 & \boxed{3} & -2 \end{pmatrix}$,

$$B = \begin{pmatrix} \boxed{1} & -1 & 7 \\ 3 & \boxed{-2} & 5 \end{pmatrix},$$

then $A+B = \begin{pmatrix} 3 & 0 & 6 \\ 3 & 1 & 3 \end{pmatrix}$.

If $C = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, $D = \begin{pmatrix} 0 & 2 \\ 0 & -3 \end{pmatrix}$,

then $C+D = \begin{pmatrix} 1 & 3 \\ 2 & 0 \end{pmatrix}$.

Note that in order to define the sum of A and B they need to be both $m\times n$

matrices for some $m, n \geq 1$.
For example, if $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & -2 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$,
then $A + C$ is not defined (i.e., we cannot add A and C).

The <u>zero $m \times n$ matrix</u> is the matrix

$$\mathbf{0}_{m\times n} = \mathbf{0} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

For any $m \times n$ matrix A, we have

$$A + \mathbf{0} = \mathbf{0} + A = A.$$

The following properties hold for any $m \times n$ matrix A, B, C:

(i) $A + (B + C) = (A + B) + C$.

(ii) $A + B = B + A$.

(iii) $A + \mathbf{0} = \mathbf{0} + A = A$.

II Multiplication of a Matrix by a Scalar

Suppose that $A = (a_{ij})_{\substack{i \leq m \\ j \leq n}}$ is an $m \times n$ matrix and $\lambda \in \mathbb{R}$ is a scalar.
We define the product $\lambda \cdot A$ to be

the matrix

$$\lambda \cdot A = (\lambda \cdot a_{ij})_{\substack{i \le m \\ j \le n}} .$$

E.g. (i) If $A = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$, then

$$2 \cdot A = \begin{pmatrix} 4 & 0 \\ -2 & 6 \end{pmatrix} .$$

(ii) If $B = \begin{pmatrix} 1 & 1 & 3 \\ -1 & -2 & 5 \end{pmatrix}$, then

$$-3B = \begin{pmatrix} -3 & -3 & -9 \\ 3 & 6 & -15 \end{pmatrix}$$

(iii) If $C = \begin{pmatrix} 1 & 5 \\ 1 & 2 \\ 0 & -1 \end{pmatrix}$, then

$$\frac{1}{2} \cdot C = \begin{pmatrix} 1/2 & 5/2 \\ 1/2 & 1 \\ 0 & -1/2 \end{pmatrix}$$

The following properties hold for any $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$:

(i) $\lambda \cdot (A + B) = \lambda A + \lambda B$

(ii) $(\lambda_1 + \lambda_2) \cdot A = \lambda_1 A + \lambda_2 A$

(iii) $\lambda_1 (\lambda_2 A) = (\lambda_1 \cdot \lambda_2) \cdot A$.

For any $m \times n$ matrix A, the matrix $-A$ is called the <u>opposite matrix</u> of A and satisfies

$$A - A = A + (-A) = \mathbf{0}.$$

E.g. when $A = \begin{pmatrix} 1 & 2 & -3 \\ 3 & -2 & 1 \end{pmatrix}$, then the opposite matrix of A is

$$-A = \begin{pmatrix} -1 & -2 & 3 \\ -3 & 2 & -1 \end{pmatrix}$$

and $A - A = \mathbf{0}_{2\times 3}$.

$$\lambda \cdot A = A \cdot \lambda$$