Lecture 1

1. Integration

Recall the fundamental theorem of calculus: If f is continuous on $[a,b]$, then the function F defined by

$$\underbrace{F(x)}_{\text{"antiderivative of } f\text{"}} = \int_a^x f(u)\,du \quad , \quad a \le x \le b \, ,$$

is continuous on $[a,b]$ and differentiable on (a,b), with

$$\frac{d}{dx} F(x) = f(x) \quad \left(\Longleftrightarrow F'(x) = f(x) \right).$$

Furthermore,

$$\int_a^b f(x)\,dx = F(b) - F(a) \, .$$

Indefinite integral:

$$\int f(x)\,dx = F(x) + \underbrace{C}_{\text{"constant"}}$$

Properties:

Properties:

(i) $\int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$

(ii) $\int \lambda \cdot f(x)\, dx = \lambda \cdot \int f(x)\, dx$.

Indefinite integrals of certain functions:

$$\int 1\, dx = x + c$$

$$\int x^{\alpha}\, dx = \frac{x^{\alpha+1}}{\alpha+1} + c$$

$$\int \frac{1}{x}\, dx = \ln|x| + c$$

$$\int e^{x}\, dx = e^{x} + c$$

$$\int \alpha^{x}\, dx = \frac{\alpha^{x}}{\ln \alpha} + c, \quad \alpha > 0, \ \alpha \neq 1$$

$$\int \sin x\, dx = -\cos x + c$$

$$\int \cos x\, dx = \sin x + c$$

$$\int \frac{1}{\cos^2 x}\, dx = \tan x + c, \quad \tan x = \frac{\sin x}{\cos x}$$

$$\int \frac{1}{\sin^2 x}\, dx = -\underbrace{\cot}_{\text{cotangent}} x + c, \quad \cot x = \frac{\cos x}{\sin x} = \frac{1}{\tan x}$$

$$\int \frac{1}{1+x^2}\, dx = \arctan x + c$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + c$$

Substitution Rule for Indefinite Integrals

Let f, g be any differentiable functions. Recall the chain rule for differentiation:

$$\big(f(g(x))\big)' = f'(g(x)) \cdot g'(x).$$

Therefore, for indefinite differentiation, we have:

$$(1) \quad \boxed{\int f(u)\,du = \int f(g(x)) \cdot g'(x)\,dx.}$$

"Proof": let $u = g(x) \Rightarrow \frac{du}{dx} = g'(x)$

Then, the right-hand side of (1) becomes

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(\underbrace{g(x)}_{=u})\,du$$

$$= \int f(u)\,du$$

$$= F(u) + c, \quad \text{where } F'(u) = f(u)$$

$$= F(g(x)) + C.$$

$=$ antiderivative of $f(g(x)) \cdot g'(x)$.

Examples:

(i) Evaluate $\int \underbrace{(2x+1)}_{= g'(x)} e^{x^2+x} \, dx$. $\quad x^2+x = g(x)$

Set $\boxed{u = g(x) = x^2 + x}$ $\Rightarrow \dfrac{du}{dx} = 2x+1$

$\Rightarrow du = (2x+1)\,dx$

$f(x) = e^x$

$\int f(g(x)) \cdot g'(x)\,dx = \int e^{x^2+x} \cdot (2x+1)\,dx$

$= \int f(u)\,du = \int e^u\,du$

$= e^u + c$

$= e^{x^2+x} + c$

(ii) Evaluate $\int \dfrac{1}{x \ln|x|}\,dx$.

Set $u = g(x) = \ln|x|$ $\Rightarrow \dfrac{du}{dx} = \dfrac{1}{x}$

$\Rightarrow du = \dfrac{1}{x}\,dx$

$f(x) = \dfrac{1}{x}$

$$\int \underbrace{\frac{1}{\ln|x|}}_{f(g(x))} \cdot \underbrace{\frac{1}{x}}_{=g'(x)} dx = \int f(u)\, du$$

$$= \ln|u| + c$$

$$= \ln\left|\ln|x|\right| + c$$

The previous two examples can be generalized as follows:

$$\int g'(x) e^{g(x)} dx = e^{g(x)} + c$$

$$\int \frac{g'(x)}{g(x)} dx = \ln|g(x)| + c.$$

this is not the derivative of x^2+1. But: we can write: $\int 4 \ldots = 2 \cdot \int 2 \ldots$

(iii) Evaluate $\int 4x \sqrt{x^2+1}\, dx$

Set $u = x^2 + 1 = g(x) \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$

$f(x) = \sqrt{x}$

$$\int \underbrace{4x}_{=2\cdot g'(x)} \underbrace{\sqrt{x^2+1}}_{=f(g(x))}\, dx = 2 \int 2x \sqrt{x^2+1}\, dx$$

$$= 2 \cdot \int f(u)\, du$$

$$= 2 \cdot \int \underbrace{\sqrt{u}}_{u^{1/2}}\, du = 2 \cdot \frac{2}{3} \cdot u^{3/2} + c$$

$$= \frac{4}{3} (x^2+1)^{3/2} + c$$

(iv) Evaluate $\int \tan x\, dx$.

We know: $\tan x = \frac{\sin x}{\cos x}$

We set: $u = g(x) = \cos x \implies \frac{du}{dx} = -\sin x$

$\implies du = -\sin x \, dx$

$f(x) = \frac{1}{x}$

$$\int \tan x \, dx = \int \underbrace{\frac{1}{\cos x}}_{= f(g(x))} \cdot \underbrace{\sin x}_{= -g'(x)} \, dx = -\int f(u) \, du$$

$$= -\int \frac{1}{u} \, du = -\ln|u| + c$$

$$= -\ln|\cos x| + c$$

(v) Evaluate $\int x \sqrt{2x-1} \, dx$.

Set $\underbrace{u = 2x-1}$ $\implies \frac{du}{dx} = 2 \implies dx = \frac{1}{2} du$

$u+1 = 2x$

$x = \frac{u+1}{2}$

Then, we have:

$$\int x \sqrt{2x-1} \, dx = \int \frac{u+1}{2} \sqrt{u} \cdot \frac{1}{2} du$$

$$= \frac{1}{4} \int \left(u^{3/2} + \sqrt{u}\right) du$$

$$= \frac{1}{4} \cdot \left(\frac{2}{5} \cdot u^{5/2} + \frac{2}{3} u^{3/2} \right) + c$$

$$= \frac{1}{10} u^{5/2} + \frac{1}{6} u^{3/2} + c$$

$$= \frac{1}{10} (2x-1)^{5/2} + \frac{1}{6} (2x-1)^{3/2} + c$$

Alternative solution: Try to make the substitution $u = \sqrt{2x-1}$ for the same integral.

(vi) let $g(x)$ be a differentiable function whose derivative $g'(x)$ is continuous.

Evaluate $\int g'(x)\cos(g(x))\,dx$.

Set $u = g(x) \Rightarrow \frac{du}{dx} = g'(x) \Rightarrow du = g'(x)\,dx$

$f(x) = \cos(x)$

$$\Rightarrow \int g'(x)\cos(g(x))\,dx = \int f(u)\,du$$

$$= \int \cos u\,du = \sin u + C$$

$$= \underline{\underline{\sin(g(x)) + C}}$$

Substitution Rule for Definite Integrals

We have:

$$\boxed{\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du},$$

because, by setting $u = g(x) \Rightarrow \frac{du}{dx} = g'(x)$

$\Rightarrow \underline{\underline{du = g'(x)\,dx}}$, we get

$$\int_a^b f(g(x))\cdot \underline{\underline{g'(x)\,dx}} = \int_{g(a)}^{g(b)} f(u)\,du,$$

and, for $x = a$, we get $u(a) = g(a)$ as lower integration bound, and $u(b) = g(b)$ as upper integration bound.

Examples:

(i) Evaluate $\int_0^4 2x\sqrt{x^2+1}\,dx$.

Set $u = x^2+1 = g(x) \Rightarrow \frac{du}{dx} = 2x \Rightarrow du = 2x\,dx$

$f(x) = \sqrt{x}$

$$\int_0^4 2x\sqrt{x^2+1}\,dx = \int_{g(0)=1}^{g(4)=17} f(u)\,du = \int_1^{17} u^{1/2}\,du$$

$$= \frac{2}{3}\left[u^{3/2}\right]_1^{17} = \underline{\underline{\frac{2}{3}\left(17^{3/2}-1\right)}}$$

(ii) Compute $\int_1^2 \frac{3x^2+1}{x^3+x}\,dx$

Set $u = x^3+x = g(x) \Rightarrow \frac{du}{dx} = 3x^2+1$

$\Rightarrow du = (3x^2+1)\,dx$

$f(x) = \frac{1}{x}$

Then $\int_1^2 \frac{3x^2+1}{x^3+x}\,dx = \int_{g(1)=2}^{g(2)=10} f(u)\,du = \int_2^{10} \frac{1}{u}\,du$

$$= \left[\ln|u|\right]_2^{10} = \ln 10 - \ln 2 = \ln\frac{10}{2}$$

$$= \underline{\underline{\ln 5}}$$

(iii) Evaluate $\int_{1/2}^1 \frac{1}{x^2} e^{1/x}\,dx$.

Set: $u = \frac{1}{x} = x^{-1} = g(x) \Rightarrow \frac{du}{dx} = -x^{-2}$

$\Rightarrow du = -x^{-2}\,dx = -\frac{1}{x^2}\,d$

$f(x) = e^x$.

We have: $\int_{1/2}^1 \frac{1}{x^2} e^{1/x}\,dx = -\int_{\underline{\underline{g(\frac{1}{2})=2}}}^{\underline{\underline{g(1)=1}}} f(u)\,du$

$$= \int_1^2 f(u)\,du = \int_1^2 e^u\,du = \left[e^u\right]_1^2$$

$= e^2 - e$

Note that because $u = \frac{1}{x}$ is a decreasing function for $x > 0$, the lower limit of integration is greater than the upper limit of integration after the substitution. It holds:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

(iv) Find $\int_0^{\pi/6} \cos x \; e^{\sin x}\,dx$

We substitute: $u = \sin x$ $\quad \frac{du}{dx} = \cos x \Rightarrow du = \cos x\,dx$

$f(x) = e^x$

limits: $x = 0 \Rightarrow u = \sin(0) = 0$

$x = \frac{\pi}{6} \Rightarrow u = \sin\frac{\pi}{6} = \frac{1}{2}$

$$\int_0^{\pi/6} \cos x \; e^{\sin x}\,dx = \int_0^{1/2} e^u\,du = \left[e^u\right]_0^{1/2}$$

$$= e^{1/2} - e^0 = \underline{\underline{e^{1/2} - 1}}$$