Exam MA0002 - Mathematical Methods B - Spring 2023

Problem 1 [30 points]

a) Evaluate the integral

$$\int_0^1 \frac{x^2}{(x^3+2)^3}dx.$$

b) We are given the following matrices:

$$A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 4 \\ 2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -3 \\ -2 & 4 \end{pmatrix}.$$

1. For each of the matrices A, B and C: Determine if the matrix is invertible or not.
2. Calculate $A + B^{-1}$.
3. Given the vectors $a = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ -\frac{3}{2} \end{pmatrix}$. Are a and b orthogonal? What does this mean graphically? Illustrate the two vectors.

Solution:

a) 10 points We make the substitution

$$u = x^3 + 2 \quad \Longrightarrow \quad \frac{du}{dx} = 3x^2 \quad \Longrightarrow \quad dx = \frac{du}{3x^2}.$$

We get

$$\begin{aligned}
\int_0^1 \frac{x^2}{(x^3+2)^3}dx &= \int_{u(0)}^{u(1)} \frac{x^2}{u^3}\frac{du}{3x^2} \\
&= \frac{1}{3}\int_{u(0)=2}^{u(1)=3} \frac{1}{u^3}du \\
&= \frac{1}{3}\left[\left(-\frac{1}{2}\right)\cdot\frac{1}{u^2}\right]_2^3 \\
&= -\frac{1}{6}\left(\frac{1}{3^2} - \frac{1}{2^2}\right) = -\frac{1}{6}\left(\frac{1}{9} - \frac{1}{4}\right) \\
&= -\frac{1}{6}\left(\frac{4}{36} - \frac{9}{36}\right) = -\frac{1}{6}\left(-\frac{5}{36}\right) \\
&= \frac{5}{216} \approx 0.0231.
\end{aligned}$$

b)

1. 6 points Which of the matrices A, B and C is invertible?

$$\det(A) = 2 \cdot 2 - 1 \cdot 4 = 0 \quad \Longrightarrow \quad A \text{ is not invertible}$$
$$\det(B) = (-1) \cdot 0 - 2 \cdot 4 = -8 \quad \Longrightarrow \quad B \text{ is invertible}$$
$$\det(C) = 1 \cdot 4 - (-3) \cdot (-2) = -2 \quad \Longrightarrow \quad C \text{ is invertible.}$$

2. 4 points Calculate $A + B^{-1}$. We have

$$B^{-1} = \frac{1}{8} \begin{pmatrix} 0 & 4 \\ 2 & 1 \end{pmatrix}$$

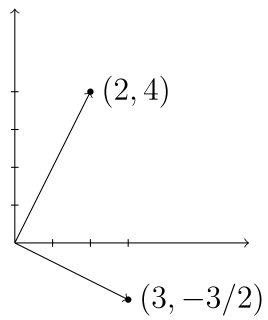
and hence

$$A + B^{-1} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} + \frac{1}{8} \begin{pmatrix} 0 & 4 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3/2 \\ 17/4 & 17/8 \end{pmatrix}$$

3. 10 points We have for the scalar (dot) product

$$a \cdot b = a^T b = 2 \cdot 3 + 4 \cdot \left(-\frac{3}{2}\right) = 0.$$

This means that a and b are orthogonal, hence, the two vectors are rectangular (i.e., they form a 90 degree angle):



Problem 2 [20 points] Solve the linear system

$$\begin{aligned} x + y + z &= 1 \\ 4x + 2y + z &= 13 \\ 6x + 4y + 3z &= 15 \end{aligned}$$

by giving a formula for *all* its solutions.

Solution:

We can see that the third equation is equal to the first equation times two plus the second equation, and thus, we can ignore the third equation. Since we have three unknowns and two equations, there will be infinitely many solutions. 5 points

Multiplying the first equation by -4 and adding it to the second, we obtain:

1	1	1	1
4	2	1	13
1	1	1	1
0	-2	-3	9.

This yields the equations

$$\begin{aligned} &x + y + z = 1 \\ &-2y - 3z = 9. \end{aligned}$$ 5 points

Let, for example, $z = t$ (alternatively, we could set x or y equal to t). Then

$$-2y - 3t = 9,$$

which gives

$$y = \frac{3t + 9}{-2}.$$

Plugging this term into the first equation, we obtain

$$x + \frac{3t + 9}{-2} + t = 1,$$

and hence

$$x = 1 - \frac{3t + 9}{-2} - t = 1 + \frac{1}{2}t + \frac{9}{2}.$$

Therefore, the solutions are of the form

$$\left(1 + \frac{1}{2}t + \frac{9}{2}, \frac{3t + 9}{-2}, t\right).$$ 10 points

Problem 3 [25 points] Suppose that

$$L = \begin{pmatrix} 0 & 2 \\ 0.5 & 0 \end{pmatrix}$$

is the Leslie matrix for a population.

a) What information does the matrix L provide about the population? (age groups, offspring rate, survival rates)

b) Suppose that at time t, the population consists of 200 0-year olds and 190 1-year olds. How many 0- and 1-year olds will be there after one year (i.e., after one reproductive cycle)?

c) Determine both eigenvalues of L.

d) Give a biological interpretation of the larger eigenvalue of L.

e) Find the stable age distribution of the population.

Solution:

a) There are two age groups (0-years olds and 1-year olds).

$$L = \begin{pmatrix} F_0 & F_1 \\ P_0 & 0 \end{pmatrix},$$

$F_0 = 0$ is the offspring rate of the 0-year olds; it means that the 0-year olds do not produce any offspring. $F_1 = 2$ is the offspring rate of the 1-year olds; it means that the 1-year olds produce on average 2 offspring. $P_0 = 0.5$ is the survival rate of the 0-year olds; it means that on average, 50% of 0-year olds survive. 3 points

b) We have $N_0(t) = 200$ and $N_1(t) = 190$.

$$N(t+1) = L \cdot N(t) = \begin{pmatrix} 0 & 2 \\ 0.5 & 0 \end{pmatrix} \cdot \begin{pmatrix} 200 \\ 190 \end{pmatrix} = \begin{pmatrix} 380 \\ 100 \end{pmatrix}.$$

This means that after one year, there will be 380 0-year olds and 100 1-year olds. 2 points

c) In order to determine the eigenvalues, we solve

$$\det(L - \lambda I_2) = 0,$$

hence

$$\det\left(\begin{pmatrix} 0 & 2 \\ 0.5 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right) = \det\left(\begin{pmatrix} -\lambda & 2 \\ 0.5 & -\lambda \end{pmatrix}\right) = \lambda^2 - 2 \cdot 0.5 = \lambda^2 - 1 = 0,$$

which gives the eigenvalues

$$\lambda_1 = 1, \quad \lambda_1 = -1.$$ 8 points

d) The larger eigenvalue $\lambda_1 = 1$ corresponds to the asymptotic growth rate of the population. After sufficiently many years, the population at time $t+1$ will be approximately the same (λ_1-times) as the population at time t. 2 points

e) We know that the eigenvector which corresponds to the larger eigenvalue is a stable age distribution. Let us compute an eigenvector that corresponds to the larger eigenvalue $\lambda_1 = 1$:

$$Lu = \lambda_1 \cdot 1 = \begin{pmatrix} 0 & 2 \\ 0.5 & 0 \end{pmatrix} \cdot \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

which gives the equation

$$2u_2 = u_1.$$

Any vector $u = (u_1, u_2)^T$ satisfying the equation $2u_2 = u_1$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 1$. For example, one eigenvector is

$$u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

This means that after a large number of reproductive sessions, the fraction of 0-year olds in the population is $\frac{2}{3}$, and the fraction of 1-year olds in the population is $\frac{1}{3}$. 10 points

Problem 4 [25 points] Researchers have observed the growth of a mice population. They concluded that the mass of the entire population $N(t)$ in kilogram at time $t > 0$ is given by the differential equation

$$\frac{dN(t)}{dt} = N(t)\left(1 - \frac{N(t)}{2}\right).$$

a) Find $N(t)$ given that the mass of the population is $N(0) = 1$ (kilogram) at time $t = 0$.

b) What is the carrying capacity of the population?

c) Find the equilibrium points of the population growth model described above and classify them according to their stability.

d) Some researchers that studied the mass of the population of mice have suggested that the growth model described above is not realistic for the following reason: it does not take into account the fact that when the mass of then population $N(t)$ is very small, the population will decline because of the lack of mates for reproduction or because lightweight mice are inclined to sickness. Can you suggest a variant of the given differential equation that takes into account the fact that when the mass of the population of mice is less than 1/2, then the mass of the population will decrease to 0? (You do not need to solve this differential equation.)

Solution:

a)

We solve the differential equation

$$\frac{dN}{dt} = N\left(1 - \frac{N}{2}\right)$$

by separation of variables:

$$\frac{dN}{N\left(1 - \frac{N}{2}\right)} = \frac{dN}{N - \frac{N^2}{2}} = \frac{dN}{\frac{2N - N^2}{2}} = \frac{2dN}{N(2-N)} = dt.$$

By partial fraction decomposition, we get

$$\frac{2}{N(2-N)} = \frac{1}{N} - \frac{1}{N-2},$$

and hence, we obtain

$$\frac{2dN}{N(2-N)} = \left(\frac{1}{N} - \frac{1}{N-2}\right) dN = dt.$$

Therefore,

$$\int \left(\frac{1}{N} - \frac{1}{N-2}\right) dN = \int dt.$$

We get

$$\ln N - \ln(N-2) = t + \tilde{C}, \quad \tilde{C} \text{ constant}.$$

Using the rules for the logarithm, we get

$$\ln N - \ln(N-2) = \ln \frac{N}{N-2}.$$

Hence,

$$\frac{N}{N-2} = e^t \cdot C, \quad C \text{ constant}.$$

Using the initial condition $N(0) = 1$, we get

$$\frac{1}{1-2} = e^0 \cdot C,$$

which gives $C = -1$. Therefore,

$$\frac{N}{N-2} = -e^t \quad \Longrightarrow \quad N = -(N-2)e^t = -Ne^t + 2e^t$$

$$\Longrightarrow \quad N + Ne^t = N(1+e^t) = 2e^t$$

and hence

$$N(t) = \frac{2e^t}{1+e^t}.$$ 13 points

b)

The given differential equation is a logistic differential equation, i.e. of the form

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right),$$

with $r = 1$ and $K = 2$. The carrying capacity of the population is $K = 2$. 2 points

c) We solve

$$g(N) := N\left(1 - \frac{N}{2}\right) = N - N^2/2 = 0$$

and get $N_1 = 0$ and $N_2 = 2$.

$$g'(N) = 1 - N,$$

hence

$$g'(0) = 1 > 0 \implies N_1 = 0 \text{ is an unstable equilibrium.}$$
$$g'(2) = 1 - 2 = -1 < 0 \implies N_2 = 2 \text{ is a locally stable equilibrium.}$$ 8 points

d)

We can propose

$$\frac{dN}{dt} = N\left(1 - \frac{N}{2}\right) \cdot \left(N - \frac{1}{2}\right).$$ 2 points