Norwegian University of Science and Technology
Department of Mathematical Sciences

Mathematical Methods B
Spring 2022

Exercise set 12

Difficulty level: Easier

1 Solve the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\cos x = 2x\mathrm{e}^{-\sin x}, \qquad y(\pi) = 0.$$

2 Assume that the size of a population evolves according to the logistic equation with intrinsic rate of growth $r = 1.5$. Assume that the carrying capacity $K = 100$.

(a) Find the differential equation that describes the rate of growth of this population.

(b) Find all equilibria, and, using the graphical approach, discuss the stability of the equilibria.

(c) Find the eigenvalues associated with the equilibria, and use the eigenvalues to determine the stability of the equilibria. Compare your answers with your results in (b)

3 Assume that the size of a population evolves according to the logistic equation with intrinsic rate of growth $r = 2$. Assume that $N(0) = 10$.

(a) Determine the carrying capacity K if the population grows fastest when the population size is 1000. *Hint*: Show that the graph dN/dt as a function of N has a maximum at $K/2$.

(b) If $N(0) = 10$, how long will it take the population size to reach 1000?

(c) Find $\lim_{t\to\infty} N(t)$.

Difficulty level: Medium

4 This problem is about finding an expression for the temperature in the water of two different thermo mugs (Sarek and Eva Solo) as functions of time (in hours) after filling with boiling water. The water temperature of the two mugs are measured at several times after filling (see Table 1). The temperature in the room were the mugs were placed was $24,2\,°\mathrm{C}$.

Time after filling (timer)	Water temperature Sarek (°C)	Water temperature Eva Solo (°C)
0	94.3	94.3
1	92.6	89.0
2	90.3	85.8
3	88.9	83.7
5	85.9	78.7
7	82.9	74.6
11	77.8	67.8
12	76.5	66.5
53	47.5	37.0
75	39.9	32.1
96	34.8	28.9
192	24.2	24.2

Table 1: Data for the two thermo mugs

Newton's law of cooling states that if we place an object in a room of constant temperature, the rate of change of the object's temperature will be proportional to the difference between the temperature of the object and the temperature in the room.

(a) Use the given information to set up a differential equation representing the relationship between the rate of change of temperature and the current temperature of the two thermo mugs. Solve the equations.

(b) Draw the graphs of the functions in (a) in the same coordinate system (possibly using Geogebra). Compare the results and comment.

(c) Evaluate the models you have found by comparing with the data from the measurements.

Difficulty level: More challenging

5 Refer to the simple model for an epidemics in subsection 8.3.1 in the textbook.

(a) Divide equation (8.75) by (8.74) to show that when $I > 0$, then:

$$\frac{\mathrm{d}I}{\mathrm{d}S} = \frac{a}{b}\frac{1}{S} - 1.$$

Also, show that when $R(0) = 0$, $I(0) = I_0$, and $S(0) = S_0$, the solution of (8.84) satisfies

$$I(t) = N - S(t) + \frac{a}{b}\ln\frac{S(t)}{S_0},$$

where $I(t)$ denotes the number of infectives, N the total number of individuals in the population, and $S(t)$ the number of susceptibles at t.

(b) We have $\mathrm{d}I/\mathrm{d}t = bSI - aI$, such that i $S(0) > a/b$, then $\mathrm{d}I/\mathrm{d}t > 0$. But since $\lim_{t\to\infty} I(t) = 0$, there exists a $t > 0$ where $I(t)$ is maximal.

Show that the number of susceptibles (S) when $I(t)$ is maximal is $S = a/b$.

(c) In (a) we found the expression $I(t)$ as a function of $S(t)$. Use the result from part (b) to show that the maximum number of infectives is given by

$$I_{max} = N - \frac{a}{b} + \frac{a}{b} \ln \frac{a/b}{S_0}$$

(d) Use the result in part (c) to show that I_{max} is a decreasing function with respect to $\frac{a}{b}$ når $\frac{a}{b} < S_0$. Use this to explain how a og b describes the severity of the disease.

Does this make sense?

Deadline: Moday, April 25th, 2022 (digitally as a single pdf-file via Blackboard)