Norwegian University of Science and Technology
Department of Mathematical Sciences

Mathematical Methods B
Spring 2022

Exercise set 11

Warm-up exercises (optional, not to be submitted)

1 Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - 2y = 3$$

(a) as a first order linear differential equation.

(b) as a seperable differential equation.

(c) Find the unique solution to the differential equation which has a graph going through $(x, y) = (0, -1)$.

2 Solve the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y\cos x = 2x\mathrm{e}^{-\sin x}, \qquad y(\pi) = 0.$$

Difficulty level: Easier (mere calculations)

3 Show that

$$y(t) = \frac{M}{1 - \left(1 - \frac{M}{y_0}\right) \cdot \mathrm{e}^{M \cdot k \cdot t}}$$

is a solution to the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k \cdot y(t) \cdot (y(t) - M)\ , \quad y(0) = y_0$$

for the constants $k < 0$, $M > 0$.

8.1.34 Solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (3 - y(x))(2 + y(x)).$$

Difficulty level: Medium (Use content from another chapter, recommended as preparation for the exam

5 Show that

$$u(t,x) = \frac{1}{\sqrt{t}} \cdot \mathrm{e}^{-\frac{x^2}{4\cdot t}}$$

solves the socalled diffusion equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2}\,.$$

Sketch the solution for a fixed value of $t > 0$ as a function of x.

Hint: The diffusion equation is «read» as an ordinary differential equation: «Ordinary» derivatives «$\frac{\mathrm{d}y(t)}{\mathrm{d}t}$» are here replaced by partial derivatives.

Difficulty level: Challenging (More aspects combined)

6 Given the initial value problem

$$\frac{\mathrm{d}y(x)}{\mathrm{d}x} = 2 \cdot x \cdot y(x)\,, \quad y(0) = 1\,.$$

(a) Find the solution $y(x)$.

(b) Use the fundamental theorem of calculus to find an expression

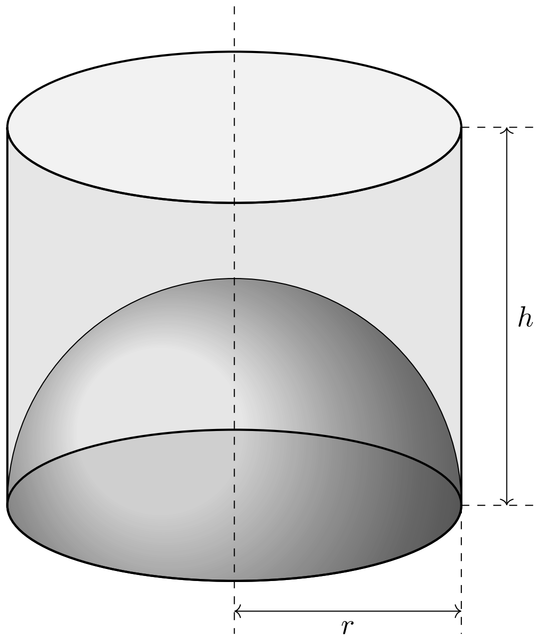
$$y(x) = \begin{matrix}\text{«something that can contain an integral from } t = 0 \text{ to } x \\ \text{over something that might be dependendt upon } y(t)\text{»}\end{matrix}\,.$$

(c) Assume we want to decide the solution *iteratively*, that is, by starting with a rough guess that will be improved for each iteration. We start with $y(x) \approx y_0(x) = 1+x$. Use the result from (b) and $y_0(x)$ to compute a new approximation (namely $y_1(x)$). Then use y_1 and (b) to compute an even better approximation $y_2(x)$.

(d) Sketch the solution $y(x)$ together with the approximations $y_0(x)$, $y_1(x)$, og $y_2(x)$ in a coordinate system for $x \in [-1, 1]$.

Hint: The method we are (re)inventing here is called «Picard iteration».

7 The task here is to decide the radius r and the height h such that the volume of the cylinder-ish container is as large as possible when the surface is a fixed positive number A (Note that you should only write the system, you do not need to solve it). The peculiarity of this cylinder (see the figure below) is that the bottom is a half sphere. Use Lagrange multipliers to find the system of equations.



Deadline: Sunday, April 3, 2022 (digitally as a single pdf-file via Blackboard)