NTNU

Kunnskap for en bedre verden

Institutt for matematiske fag

Eksamensoppgave i **MA0002 Brukerkurs i Matematikk II**

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Eksamenstid (fra–til): 09:00 - 12:00

Hjelpemiddelkode/Tillatte hjelpemidler: B: Alle trykte og håndskrevne hjelpemidler tillatt. Bestemt, enkel kalkulator tillatt.

Annen informasjon:
Annen info? Hvilken annen info?

Målform/språk: bokmål

Antall sider: 8

Antall sider vedlegg: 2

Kontrollert av:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☒	**farger** ☐
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Dato Sign

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Oppgave 1

A. Evaluate the integral

$$\int_0^1 \frac{x}{x^2+2}\mathrm{d}x.$$

B. Consider the matrices

$$A = \begin{pmatrix} 2 & -1 \\ -7 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 4 \\ 2 & 10 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}.$$

a) Which of the matrices A, B, C are invertible?

b) Find the matrix $B + A^{-1}$.

c) Consider the vectors $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Are $\mathbf{b}, \mathbf{c}$ orthogonal?

Solution

A. We set $u = x^2 + 2$, so that $du = 2xdx$ and the new limits of integration are $u_1 = 2, u_2 = 3$. Thus

$$\int_0^1 \frac{x}{x^2+2}\mathrm{d}x = \int_2^3 \frac{\mathrm{d}u}{2u} = \left[\frac{1}{2}\ln|u|\right]_2^3 = \frac{1}{2}\ln\left(\frac{3}{2}\right).$$

B. a) We calculate the determinants of the matrices.

$$\det A = 6 - 7 = -1 \neq 0, \quad \det B = -10 - 8 = -18 \neq 0, \quad \det C = 4 - 4 = 0.$$

Therefore A and B are invertible, while C is not invertible.

b) The inverse of A is

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} 3 & 1 \\ 7 & 2 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ -7 & -2 \end{pmatrix}.$$

Thus

$$B + A^{-1} = \begin{pmatrix} -1 & 4 \\ 2 & 10 \end{pmatrix} + \begin{pmatrix} -3 & -1 \\ -7 & -2 \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ -5 & 8 \end{pmatrix}.$$

c) We find the scalar product of $\mathbf{b}$ and $\mathbf{c}$.

$$\mathbf{b} \cdot \mathbf{c} = \begin{bmatrix} -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \end{bmatrix} = -1 - 4 = -5 \neq 0.$$

Therefore the two vectors are not orthogonal.

Oppgave 2 A group of biologists have been observing a population of lions in the jungle. They have come to the conclusion that the number of individuals $N(t)$ in the population at time $t > 0$ grows according to the differential equation

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = \frac{N(t)}{1000}\big(500 - N(t)\big).$$

a) Find $N(t)$, given that the population at time $t = 0$ is equal to $N(0) = 50$.

b) What does the differential equation imply about the per capita growth rate of the population? What is the carrying capacity of the population?

c) Find the equilibrium points of the population growth model described above and classify them according to their stability.

d) Some biologists from the group that studied the population of lions have suggested that the growth model described above is not realistic for the following reason: it does not take into account the fact that when the number $N(t)$ of lions is very small, the population will decline because of the lack of mates for reproduction. Can you suggest a variant of the given differential equation that takes into account the fact that when the population of lions is less than 40 then the population will decrease to 0? (You do not need to solve the new differential equation you will write.)

Solution

a) We solve the differential equation.

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} = \frac{N}{1000}(500 - N) &\Rightarrow \frac{\mathrm{d}N}{N(500-N)} = \frac{\mathrm{d}t}{1000} \\ &\Rightarrow \int \frac{\mathrm{d}N}{N(500-N)} = \int \frac{\mathrm{d}t}{1000} \\ &\Rightarrow \int \frac{1}{500}\Big(\frac{1}{N} + \frac{1}{500-N}\Big)\mathrm{d}N = \frac{t}{1000} + C \\ &\Rightarrow \frac{1}{500}\ln\left(\frac{N(t)}{500 - N(t)}\right) = \frac{t}{1000} + C.\end{aligned}$$

We now find the value of the constant C. At time $t = 0$ we have $N(0) = 50$, therefore

$$\frac{1}{500}\ln\Big(\frac{50}{450}\Big) = C \quad \Rightarrow \quad C = -\frac{\ln 9}{500}.$$

Substituting back the value of C, we find

$$\begin{aligned}\frac{1}{500}\ln\left(\frac{N(t)}{500-N(t)}\right)=\frac{t}{1000}-\frac{\ln 9}{500} &\Rightarrow \ln\left(\frac{N(t)}{500-N(t)}\right)=\frac{t}{2}-\ln 9\\ &\Rightarrow \ln\left(\frac{500-N(t)}{N(t)}\right)=\ln 9-\frac{1}{2}t\\ &\Rightarrow \frac{500}{N(t)}-1=9e^{-t/2}\\ &\Rightarrow N(t)=\frac{500}{1+9e^{-t/2}}.\end{aligned}$$

b) According to the differential equation, the per capita growth rate is

$$\frac{\mathrm{d}N/\mathrm{d}t}{N}=\frac{1}{1000}(500-N),$$

so it depends linearly on $N(t)$; more specific, the bigger $N(t)$ is, the smaller the per capita growth rate becomes.
The given differential equation is a logistic differential equation, i.e. of the form

$$\frac{\mathrm{d}N}{\mathrm{d}t}=rN\Big(1-\frac{N}{K}\Big),$$

with $r=0.5$ and $K=500$. The carrying capacity of the population is $K=500$.
c) We set

$$g(N)=\frac{N}{1000}(500-N).$$

The equilibrium points are the solutions of $g(N)=0$, so they are $N=0$ and $N=500$. We have

$$g'(N)=\frac{1}{2}-\frac{N}{500}.$$

Since $g'(0)=1/2>0$, $N=0$ is an unstable equilibrium point.
Since $g'(500)=-1/2<0$, $N=500$ is a locally stable equilibrium point.
d) We can suggest the differential equation

$$\frac{\mathrm{d}N}{\mathrm{d}t}=\frac{N}{1000}(N-40)(500-N).$$

Oppgave 3 The growth of a population of mice (krattspissmus) is described by the following Leslie matrix

$$L = \begin{bmatrix} 0 & 5 \\ 0.8 & 0 \end{bmatrix}.$$

a) What information does the matrix L provide about the population? (age groups, offspring rate, survival rates)

b) Find the eigenvectors and eigenvalues of L.

c) What is the biological interpretation of the larger eigenvalue of L?

d) What is the percentage of each age group in the population after a large number of reproduction sessions? Justify your answer.

Solution

a) The population (of females) consists of two age groups, the 0-year olds and the 1-year olds. On average, every year the 80% of the 0-year olds survive. Also the 0-year olds do not produce offspring, while each 1-year old female gives on average 5 mice.

b) We have

$$\det(\lambda I - L) = \begin{vmatrix} \lambda & -5 \\ -0.8 & \lambda \end{vmatrix} = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2).$$

The eigenvalues of L are therefore $\lambda_1 = 2$ and $\lambda_2 = -2$. We now find an eigenvector $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ of L corresponding to the eigenvalue $\lambda_1 = 2$.

$$L\mathbf{u} = 2\mathbf{u} \quad \Rightarrow \quad \begin{bmatrix} 2 & -5 \\ -0.8 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 2u_1 - 5u_2 = 0,$$

hence $\mathbf{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 2$.

Next, we find an eigenvector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ corresponding to $\lambda_2 = -2$.

$$L\mathbf{v} = -2\mathbf{v} \quad \Rightarrow \quad \begin{bmatrix} -2 & -5 \\ -0.8 & -2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad 2v_1 + 5v_2 = 0,$$

hence $\mathbf{v} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_2 = -2$.

c) The larger eigenvalue of L expresses the asymptotic growth rate of the population. After sufficiently many years, the population at time $t+1$ will be approximately 2 times the population at time t.

d) We know that the eigenvector $\mathbf{u}$ corresponding to the larger eigenvalue is a stable age distribution. This means that after a large number of reproduction sessions, the population of mice will consist of approximately

$$\frac{5}{7} \approxeq 71.43\% \text{ of 0-year olds}$$

and

$$\frac{2}{7} \approxeq 28.57\% \text{ of 1-year olds.}$$

Oppgave 4 Consider the function

$$f(x, y) = x^2 - 2xy + y^3$$

defined on the closed and bounded domain

$$D = \{(x, y) \in \mathbb{R}^2 : -1 \leqslant x \leqslant 2, -1 \leqslant y \leqslant 2\}.$$

a) Find the global maximum and minimum of f on the domain D.

b) Does f have a local maximum or minimum at the point $(0, 0)$? Does it have a local maximum or minimum at $(\frac{3}{2}, \frac{3}{2})$?

c) Find the equation of the tangent plane of the graph of f at the point $(1, 1)$.

d) Find the directional derivative of f at the point $(1, 1)$ towards the direction of the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solution

a) The gradient of f is

$$\nabla f(x, y) = \begin{bmatrix} 2x - 2y \\ 3y^2 - 2x \end{bmatrix}.$$

We first find the potential extremum points of f in the interior of D. These points satisfy

$$\nabla f(x, y) = \mathbf{0} \quad \Leftrightarrow \quad \begin{cases} x = y \\ 3y^2 - 2x = 0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} x = y \\ 3y^2 - 2y = 0 \end{cases}$$

hence we have the points $(x, y) = (0, 0)$ and $(x, y) = (\frac{2}{3}, \frac{2}{3})$. The values of f at these points are

$$f(0,0) = 0 \qquad \text{and} \qquad f\left(\frac{2}{3}, \frac{2}{3}\right) = -\frac{4}{27}.$$

Next, we find the values of f on the boundary of D. We name the vertices of D as

$$A(-1,\, 2), \quad B(2,\, 2), \quad \Gamma(2, -1), \quad \Delta(-1, -1),$$

and examine each of the sides separately.

<u>On AB:</u> $-1 \leqslant x \leqslant 2, \quad y = 2$.
$f(x, y) = f(x,\, 2) = x^2 - 4x + 8 =: g(x), \quad -1 \leqslant x \leqslant 2$.
$g'(x) = 2x - 4 < 0$ for all $x < 2$, so g is decreasing on $[-1, 2]$. Since $g(-1) = 13$ and $g(2) = 4$, the maximum value of $f(x, y) = g(x)$ on AB is $f(-1, 2) = 13$ and its minimum value is $f(2, 2) = 4$.

<u>On $B\Gamma$:</u> $x = 2, -1 \leqslant y \leqslant 2$.
$f(x, y) = f(2, y) = 4 - 4y + y^3 =: h(y)$, $-1 \leqslant y \leqslant 2$.
Here $h'(y) = 3y^2 - 4$, therefore h is decreasing on the interval $[-1, \frac{2\sqrt{3}}{3}]$ and increasing on $[\frac{2\sqrt{3}}{3}, 2]$. Since

$$f(2,2) = 4, \quad f\left(2, \frac{2\sqrt{3}}{3}\right) = 4 - \frac{16\sqrt{3}}{9} \approx 0.9208, \quad f(2,-1) = 7,$$

the maximum value of f on $B\Gamma$ is $f(2, -1) = 7$ and the minimum value is $f(2, \frac{2\sqrt{3}}{3}) = 4 - \frac{16\sqrt{3}}{9}$.

<u>On $\Gamma\Delta$:</u> $-1 \leqslant x \leqslant 2$, $y = -1$, so
$f(x, y) = f(x, -1) = x^2 + 2x - 1 =: \phi(x)$, $-1 \leqslant x \leqslant 2$.
Since $\phi'(x) = 2x + 2 > 0$ for all $x > -1$, we find that the maximum of f on $\Gamma\Delta$ is $f(2, -1) = 7$ and its minimum is $f(-1, -1) = -2$.

<u>On ΔA:</u> $x = -1, -1 \leqslant y \leqslant 2$.
$f(x, y) = f(-1, y) = y^3 + 2y + 1$ and arguing as in the previous cases we find that the minimum of f is $f(-1, -1) = -2$ and its maximum is $f(-1, 2) = 13$.

Comparing all values of f on the boundary of D as well as the values of f at the critical points in the interior of D we see that the global minimum of f on D is

$$f(-1, -1) = -2$$

while its global maximum on D is

$$f(-1,2) = 13.$$

b) We have seen that the points $(0,0)$ and $(\frac{2}{3}, \frac{2}{3})$ are critical points of f but not points of global extremum. However we do not know a priori whether f has a local minimum/maximum on these points.

On each of these points (x_0, y_0), we need to calculate the number

$$D := f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2.$$

we have

$$f_{xx}(x,y) = 2, \qquad f_{yy}(x,y) = 6y, \qquad f_{xy}(x,y) = -2.$$

• On the point $(0,0)$,

$$D = f_{xx}(0,0) f_{yy}(0,0) - f_{xy}(0,0)^2 = -4 < 0,$$

therefore $(0,0)$ is a saddlepoint.
• On the point $(\frac{3}{2}, \frac{3}{2})$,

$$D = f_{xx}(\tfrac{2}{3}, \tfrac{2}{3}) f_{yy}(\tfrac{2}{3}, \tfrac{2}{3}) - f_{xy}(\tfrac{2}{3}, \tfrac{2}{3})^2 = 2 \cdot 4 - 4 = 4 > 0$$

and $f_{xx}(2/3, 2/3) = 2 > 0$, therefore $(2/3, 2/3)$ is a point of local minimum.

c)At the point $(1,1)$ we have

$$f_x(1,1) = 0 \qquad \text{and} \qquad f_y(1,1) = 1.$$

The equation of the tangent plane is

$$z = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1) \Rightarrow$$
$$z = y - 1.$$

d) The vector $\mathbf{v}$ has length $\|\mathbf{v}\| = \sqrt{1^2 + 2^2} = \sqrt{5}$. The normalised vector at the direction of $\mathbf{v}$ is

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \begin{bmatrix} \sqrt{5}/5 \\ 2\sqrt{5}/5 \end{bmatrix}.$$

The gradient of f at $(1,1)$ is $\nabla f(1,1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so the directional derivative at $(1,1)$ at the direction of $\mathbf{u}$ is

$$D_{\mathbf{u}} f(1,1) = \nabla f(1,1) \cdot \mathbf{u} = \frac{2\sqrt{5}}{5}.$$

Dette er en vedleggsside.

Og dette er en til.