MA0002

Mathematical Methods B

Spring 2015

Exam Solutions

Problem 1:

Compute the following integrals.

a) $\int xe^{x^2}\,dx$.

Solution: We apply u-substitution.

Let $u = x^2$. Then $du = 2x\,dx$, so

$$\int xe^{x^2}\,dx = \int \frac{1}{2}e^u\,du = \frac{1}{2}e^u + C$$
$$= \frac{1}{2}e^{x^2} + C.$$

b) $\int \frac{dx}{x(x-1)}$

Solution: We can use a partial fraction decomposition to rewrite the integral as

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

$$= \frac{A(x-1) + Bx}{x(x-1)}.$$

$$= \frac{(A+B)x - A}{x(x-1)}.$$

It follows that: $A+B=0$

$A = -1$.

Solving for B: $B = 1$. So we can write:

$$\int \frac{dx}{x(x-1)} = \int \frac{dx}{x-1} - \int \frac{dx}{x}$$

$$= \ln|x-1| - \ln|x| + C$$

$$= \ln\left|\frac{x-1}{x}\right| + C$$

c) $\int (x+1)\sin(x)\,dx$.

Solution: We apply integration by parts.

Let $u = x+1$ $\quad dv = \sin(x)$

$du = dx$ $\quad v = -\cos(x)$

Then: $\int (x+1)\sin(x)dx = -(x+1)\cos(x) - \int -\cos(x)\,dx$

Simplifying:

$$\int (x+1)\sin(x)\,dx = -(x+1)\cos(x) + \int \cos(x)\,dx$$

$$= -(x+1)\cos(x) + \sin(x) + C$$

Problem 2

Consider the function $f(x,y) = xy - 2y^2$.

a) Find the critical points of $f(x,y)$ and determine whether they are local max, min, or saddle points.

Solution: Recall that critical pts occur where $\nabla f = \vec{0}$, where $\vec{0}$ is the zero vector. So, we first compute ∇f:

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} y \\ x - 4y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives us the equations

$$y = 0$$
$$x - 4y = 0,$$

whose only solution is $x = 0$, $y = 0$. So the only critical point is $(0,0)$.

To determine if it is a max/min/saddle point, we use the following test: let

$$D = f_{xx}(0,0)\, f_{yy}(0,0) - \left(f_{xy}(0,0)\right)^2.$$

You can check that

$f_{xx}(0,0) = 0, \quad f_{yy}(0,0) = -4, \quad f_{xy}(0,0) = 1.$

So $D = -1 < 0$. By the Theorem on page 548 of the textbook, $(0,0)$ is a saddle point.

b) Find the global max and min for f on the closed set

$$D = \left\{ (x,y) : 0 \leq x \leq 1, \ 0 \leq y \leq 1 \right\}.$$

Solution: Recall that for $f(x,y)$ defined on a closed set, f must have a max and a min, which will occur either at a critical point or on the boundary of D.
In part (a), we saw that the only critical point is $(0,0)$, which is on the boundary. So it suffices to check $f(x,y)$ on the boundary of D.

First boundary part: $x = 0$.

Note that $f(0,y) = -2y^2$ = function of y only.
For single variable functions, if they are defined on a closed interval $[a,b]$, the max/min

occurs either at critical points, or at the endpoints a or b. If $g(y) = -2y^2$ and $0 \le y \le 1$, then the only possibilities are at the endpoints $a = 0$, $b = 1$ (Why?)
So, possible max/min are:

$f(0,0) = 0$
$f(0,1) = -2$

Second boundary: $y = 0$.

Check that $f(x,0) = 0$ for all x.
Thus any point of the form $(x,0)$, $0 \le x \le 1$, is a candidate, with $f(x,0) = 0$.

Third boundary: $x = 1$.

Check that $f(1,y) = y - 2y^2$, with $0 \le y \le 1$.
If $h(y) = y - 2y^2$, then

$$h'(y) = 1 - 4y.$$

Setting this equal to zero, we get

$$1 - 4y = 0, \quad \text{so} \quad y = \frac{1}{4}.$$

This gives us three candidates:

$f(1, \frac{1}{4}) = \frac{1}{8}$
$f(1, 0) = 0$
$f(1, 1) = -1.$

Fourth boundary: $y = 1$.

Check that $f(x, 1) = x - 2$. If $k(x) = x - 2$, then

$$k'(x) = 1.$$

So $k(x)$ has no critical points. Thus the only candidates along this boundary are

$f(0, 1) = -2$
$f(1, 1) = -1.$

Thus we see that the max occurs at $(1, \frac{1}{4})$, and the min at $(0, 1)$.

c) Compute the directional deriv. $D_{\vec{u}}f$ of $f(x,y)$ at $(1,1)$ in the direction of $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

<u>Solution</u>: Recall that we must use unit vectors to compute directional derivatives. Thus

we normalize $\vec{u}$:

$$|\vec{u}| = \sqrt{3^2 + 2^2}$$

$$= \sqrt{9+4} = \sqrt{13}.$$

Thus we let

$$\vec{n} = \frac{1}{\sqrt{13}} \vec{u} = \frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

Then: $D_{\vec{u}} f(1,1) = \nabla f(1,1) \cdot \vec{n}$

$$= \begin{bmatrix} 1 \\ -3 \end{bmatrix} \cdot \frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$= \frac{1}{\sqrt{13}} \cdot (3-6)$$

$$= -\frac{3}{\sqrt{13}}.$$

Problem 3

If

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 1 \end{bmatrix},$$

compute AB, A^2, B^TB, and A^2B.

Solution: Using the definition of matrix multiplication, we have:

$$AB = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot 1 + 0\cdot 2 + 1\cdot 0 & 3\cdot 2 + 0\cdot 1 + 1\cdot 1 \\ 2\cdot 1 - 1\cdot 2 + 1\cdot 0 & 2\cdot 2 - 1\cdot 1 + 1\cdot 1 \\ -1\cdot 1 + 2\cdot 2 + 1\cdot 0 & -1\cdot 2 + 2\cdot 1 + 1\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 7 \\ 0 & 4 \\ 3 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3\cdot 3 + 0\cdot 2 + 1\cdot(-1) & 3\cdot 0 + 0\cdot(-1) + 1\cdot 2 & 3\cdot 1 + 0\cdot 1 + 1\cdot 1 \\ 2\cdot 3 - 1\cdot 2 + 1\cdot(-1) & 2\cdot 0 - 1\cdot(-1) + 1\cdot 2 & 2\cdot 1 - 1\cdot 1 + 1\cdot 1 \\ -1\cdot 3 + 2\cdot 2 + 1\cdot(-1) & -1\cdot 0 + 2\cdot(-1) + 1\cdot 2 & (-1)\cdot 1 + 2\cdot 1 + 1\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 2 & 4 \\ 3 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$B^T B = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 1 + 2\cdot 2 + 0\cdot 0 & 1\cdot 2 + 2\cdot 1 + 0\cdot 1 \\ 2\cdot 1 + 2\cdot 1 + 1\cdot 0 & 2\cdot 2 + 1\cdot 1 + 1\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 4 \\ 4 & 6 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 8 & 2 & 4 \\ 3 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \cdot 1 + 2 \cdot 2 + 4 \cdot 0 & 8 \cdot 2 + 2 \cdot 1 + 4 \cdot 1 \\ 3 \cdot 1 + 3 \cdot 2 + 2 \cdot 0 & 3 \cdot 2 + 3 \cdot 1 + 2 \cdot 1 \\ 0 \cdot 1 + 0 \cdot 2 + 2 \cdot 0 & 0 \cdot 2 + 0 \cdot 1 + 2 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & 22 \\ 9 & 11 \\ 0 & 2 \end{bmatrix}$$

Problem 4

a) Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} 3 & 6 \\ -1 & -4 \end{bmatrix}$$

Solution:

Eigenvalues:

$$\det(A - \lambda I) = \det \begin{bmatrix} 3-\lambda & 6 \\ -1 & -4-\lambda \end{bmatrix} \stackrel{!}{=} 0$$

$$\Rightarrow \quad (3-\lambda)(-4-\lambda) + 6 = 0$$

$$\Rightarrow -12 - 3\lambda + 4\lambda + \lambda^2 + 6 = 0$$

$$\Rightarrow \lambda^2 + \lambda - 6 = 0$$

$$\Rightarrow \quad (\lambda + 3)(\lambda - 2) = 0$$

$$\lambda = -3, 2$$

Eigenvectors: $A\vec{v} = \lambda \vec{v}$

For $\lambda = -3$, we have:

$$\begin{bmatrix} 3 & 6 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -3v_1 \\ -3v_2 \end{bmatrix}$$

$$\left.\begin{aligned} 3v_1 + 6v_2 &= -3v_1 \\ -1v_1 - 4v_2 &= -3v_2 \end{aligned}\right\} \qquad \begin{aligned} v_1 &= -v_2 \\ -3v_2 &= -3v_2 \end{aligned}$$

Choose $v_2 = 1$. So $v_1 = -1$

First eigenvector $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

For $\lambda = 2$, we have

$$A\vec{w} = 2\vec{w} \Rightarrow \begin{bmatrix} 3 & 6 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 2w_1 \\ 2w_2 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} 3w_1 + 6w_2 &= 2w_1 \\ -1w_1 - 4w_2 &= 2w_2 \end{aligned}\right\} \qquad \begin{aligned} 6w_2 &= -1w_1 \\ 6w_2 &= -1w_1 \end{aligned}$$

So choose $w_2 = 1$. Then $w_1 = -6$

2nd eigenvector: $\vec{w} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$.

b) Find the general solution to the system

$$\frac{dx_1}{dt} = 3x_1 + 6x_2$$

$$\frac{dx_2}{dt} = -x_1 - 4x_2.$$

Solution: The system can be written as

$$\frac{d\vec{x}}{dt} = A\vec{x},$$

where $A = \begin{bmatrix} 3 & 6 \\ -1 & -4 \end{bmatrix}$.

Such a system has general solution

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + c_2 e^{\lambda_2 t}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

where λ_1, λ_2 are eigenvalues and $\vec{v}, \vec{w}$ are eigenvectors. From (a), we have

$$\vec{x}(t) = c_1 e^{-3t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

Problem 5

Suppose we have a colony of bacteria whose population $P(t)$ evolves according to the logistic equation

$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{100\,000}\right)$$

Assume the colony starts off with only 1000 bacteria.

a) Find the population $P(t)$ after t hours.

Solution: Referring to pages 400-401 of your textbook, we see that the solution of the differential equation above is given by

$$P(t) = \frac{100\,000}{1 - \frac{e^{-2t}}{c}},$$

where c is a constant we must find. To find it, we use the fact that $P(0) = 1000$ (since we start with 1000 bacteria), which gives us

$$1000 = \frac{100\ 000}{1 - \frac{1}{c}}$$

$$\Rightarrow \quad 1 - \frac{1}{c} = \frac{100\ 000}{1\ 000}$$

$$1 - \frac{1}{c} = 100$$

$$\Rightarrow \quad c - 1 = 100c$$

$$\Rightarrow \quad -99c = 1$$

$$\Rightarrow \quad c = -\frac{1}{99}.$$

Thus $P(t) = \dfrac{100\ 000}{1 + 99e^{-2t}}$

b) Determine the amount of time t to reach 50000 bacteria.

<u>Solution</u>: The question is asking us to find t such that

$$P(t) = 50000.$$

From part (a), we know that at this time t, $P(t)$ will satisfy

$$50\,000 = \frac{100\,000}{1+99e^{-2t}}$$

or:

$$\frac{1}{2} = \frac{1}{1+99e^{-2t}}.$$

$$\Longrightarrow \quad 2 = 1 + 99\,e^{-2t}$$

$$\Longrightarrow \quad 1 = 99e^{-2t}$$

$$\Longrightarrow \quad \frac{1}{99} = e^{-2t}$$

$$\Longrightarrow \quad \ln \frac{1}{99} = \ln e^{-2t} = (-2t)\ln e = -2t$$

$$\Longrightarrow \quad t = -\frac{1}{2}\ln\frac{1}{99} = \frac{1}{2}\ln 99.$$