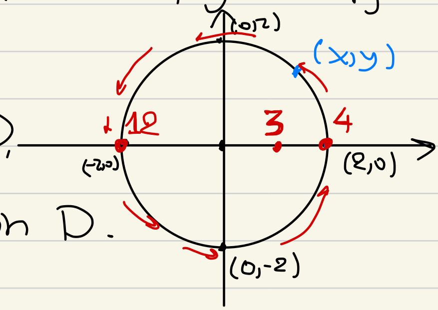
Exercise: Find the global maximum and global minimum of

$$f(x,y) = x^2 + y^2 - 2x + 4$$

on the disc $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4\}$.



- D is closed & bounded and f is continuous on D, so f has a global max. and a global minimum on D.

We find the potential extrema of $f(x,y)$ in the interior of D.

$$\nabla f(x,y) = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix} = \begin{bmatrix} 2x-2 \\ 2y \end{bmatrix}$$

$$\nabla f(x,y) = 0 \Rightarrow \begin{cases} 2x - 2 = 0 \\ 2y = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 0 \end{cases}$$

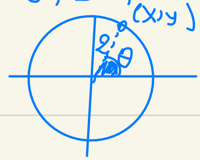
$$f(1,0) = 1^2 - 2 + 4 = 3$$

We now find the values of f on the boundary of D, which is the circle $x^2 + y^2 = 4$.

$x = 2\cos\theta$ $\quad x^2+y^2 = 4\cos^2\theta + 4\sin^2\theta = 4(\sin^2\theta + \cos^2\theta) = 4$

$y = 2\sin\theta$

(x,y) ?



We find a parametrisation of the points on the boundary of D.

$\cos\theta = \frac{x}{2}$

Points (x,y) with $x^2+y^2=4$ have coordinates of the form

$\Rightarrow x = 2\cos\theta$

$$\begin{cases} x = 2\cos\theta \\ y = 2\sin\theta \end{cases}, \quad 0 \le \theta < 2\pi.$$

$\sin\theta = \frac{y}{2} \Rightarrow$

$y = 2\sin\theta$

$\theta = 0$: $x = 2$, $y = 0$

$\theta = \frac{\pi}{2}$: $x = \sqrt{2}$, $y = \sqrt{2}$

Thus on the boundary of D,

$$\begin{aligned} f(x,y) &= f(2\cos\theta, 2\sin\theta) \\ &= 4\cos^2\theta + 4\sin^2\theta - 4\cos\theta + 4 \\ &= 8 - 4\cos\theta \\ &= 4(2-\cos\theta) \\ &\equiv g(\theta), \quad 0 \le \theta < 2\pi. \end{aligned}$$

$g(\theta)$ is maximum when

$\cos\theta = -1 \iff \theta = \pi$

and then it is equal to $g(\pi) = 12$

$g(\theta)$ is minimum when

$\cos\theta = 1 \iff \theta = 0$,

and then it is equal to $g(0) = 4$.

f has a global min. on $(1,0)$, which is 3.

f has a global max. on $(-2,0)$, which is 12.

Exercise: Find three non-negative numbers whose sum is equal to 90 such that their product is maximum.

- Let $x, y, z \geq 0$ be three non-negative numbers. Because their sum is equal to 90,

$$x + y + z = 90 \implies z = 90 - x - y.$$

Their product is

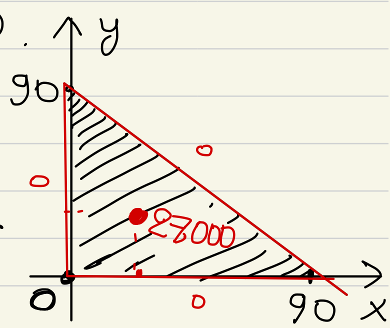
$$xyz = xy(90 - x - y).$$

We need to maximize the function

$$f(x,y) = xy(90 - x - y)$$

with $x \geq 0$, $y \geq 0$
and $z \geq 0 \implies 90 - x - y \geq 0$
$\implies x + y \leq 90$.

We find potential global extremum points in the interior of the domain of f.



$$f(x,y) = xy(90-x-y) = 90xy - x^2y - xy^2.$$

$$\nabla f(x,y) = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix} = \begin{bmatrix} 90y - 2xy - y^2 \\ 90x - 2xy - x^2 \end{bmatrix}$$

$$\nabla f(x,y) = 0 \Rightarrow \begin{cases} 90y - 2xy - y^2 = 0 \\ 90x - 2xy - x^2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (90 - 2x - y)\cdot y = 0 \\ (90 - 2y - x)\cdot x = 0 \end{cases}$$

We are looking for potential extremum points in the interior, so $x > 0$ and $y > 0$.
So the system gives

$$\begin{cases} 90 - 2x - y = 0 \\ 90 - 2y - x = 0 \end{cases} \Rightarrow \begin{cases} 2x + y = 90 \\ x + 2y = 90 \end{cases}$$

with solution $(x,y) = (30, 30)$.

$$f(30,30) = 30 \cdot 30 \cdot (90-30-30) = 27{,}000.$$

On any point (x,y) on the boundary of the domain of f, we have

$$f(x,y) = 0.$$

Hence f has a global max. at $(30,30)$ which is equal to

$$f(30,30) = 27,000.$$

The product of the three numbers is maximum when
$$x = 30, \; y = 30, \; z = 30.$$

• EXTREMA UNDER CONSTRAINTS

So far we have minimised/maximised functions $f(x,y)$ on their domains. Sometimes we need to find extrema of $f(x,y)$ under some additional constraint.

E.g. • We want to find the min. and max. of $f(x,y) = x^3 - xy + y^2$ under the constraint that
$$x^2 + y^2 = 1.$$

• We want to find the maximum & minimum of $f(x,y) = xy + 2x^2y^2$ under the constraint that
$$(x-1)^2 + 2(y-2)^2 = 2.$$

In general, we want to find the extrema of the function $f(x,y)$ under the constraint

$$g(x,y) = 0.$$

<u>THEOREM (Lagrange)</u>: Assume that f, g have continuous first-order partial derivatives and $f(x,y)$ has an extremum at (x_0, y_0) subject to the constraint $g(x,y) = 0$.

If $\nabla g(x_0, y_0) \neq 0$ then there exists some λ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

This suggests that when we want to optimise $f(x,y)$ subject to the constraint $g(x,y) = 0$, we have to solve the system:

$$\begin{cases} g(x,y) = 0 \\ \nabla f(x,y) = \lambda \cdot \nabla g(x,y) \end{cases}$$

The number λ is called a <u>Lagrange multiplier</u>.

Example: Find all extrema of

$$f(x,y) = e^{-xy}$$

subject to the constraint $x^2 + 4y^2 = 1$.

- The constraint can be written as $g(x,y) = 0$, where

$$g(x,y) = x^2 + 4y^2 - 1.$$

We solve the system

$$\begin{cases} g(x,y) = 0 \\ \nabla f(x,y) = \lambda \nabla g(x,y). \end{cases} \qquad (*)$$

$$\nabla f(x,y) = \begin{bmatrix} -y e^{-xy} \\ -x e^{-xy} \end{bmatrix}, \qquad \nabla g(x,y) = \begin{bmatrix} 2x \\ 8y \end{bmatrix}$$

$$(*) \Rightarrow \begin{cases} x^2 + 4y^2 = 1 \\ -y e^{-xy} = 2\lambda x \\ -x e^{-xy} = 8\lambda y \end{cases} \Bigg| \begin{array}{l} x^2 + 4y^2 = 1 \\ -4y^2 e^{-xy} = 8\lambda xy \\ -x^2 e^{-xy} = 8\lambda xy \end{array}$$

$$\Rightarrow \begin{cases} x^2 + 4y^2 = 1 \\ x^2 e^{-xy} = 4y^2 e^{-xy} \end{cases} \Bigg| \begin{array}{l} x^2 + 4y^2 = 1 \\ x^2 = 4y^2 \end{array}$$

$$f(x,y) = e^{-xy}$$

$$\Rightarrow \begin{cases} 2x^2 = 1 \\ x^2 = 4y^2 \end{cases} \Bigg| \begin{array}{l} x^2 = 1/2 \\ y^2 = 1/8 \end{array} \Bigg| \begin{array}{l} x = \frac{\sqrt{2}}{2} \text{ OR } x = -\frac{\sqrt{2}}{2} \\ y = \frac{\sqrt{2}}{4} \text{ OR } y = -\frac{\sqrt{2}}{4}. \end{array}$$

The system has 4 solutions:

$$(x,y) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4}\right), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4}\right) \text{ and } \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4}\right).$$

We have to find the values of $f(x,y)$ on all these points.

$$f\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4}\right) = f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4}\right) = e^{-1/4}$$

$$f\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{4}\right) = f\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4}\right) = e^{1/4}$$

The maximum value of $f(x,y)$ subject to the constraint $x^2+4y^2=1$ is equal to $e^{1/4}$.

The minimum value of $f(x,y)$ subject to the constraint $x^2+4y^2=1$ is equal to $e^{-1/4}$.

Exercise: Among all positive numbers with product equal to 1, find those whose sum is minimum.

Of course we could say that

$$xy = 1 \Rightarrow y = \frac{1}{x}$$

and the sum of the two numbers is

$$x + y = x + \frac{1}{x}.$$

So we need to find the minimum of

$$h(x) = x + \frac{1}{x}, \quad x > 0.$$

But instead we use Lagrange multipliers.

We want to minimize

$$f(x,y) = x + y, \quad x, y > 0$$

subject to the constraint $xy = 1$.

Set

$$g(x,y) = xy - 1.$$