First we find the solution mathematically.
(similar to $y' = 2(y-1)(y+2)$).

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) \Rightarrow$$

$$\int \frac{dN}{N\left(1-\frac{N}{K}\right)} = \int r\, dt \Rightarrow$$

$$\vdots$$

$$N(t) = \frac{K}{1 + \left(\frac{K}{N_0} - 1\right)e^{-rt}}.$$

We now proceed to the explanation of the logistic equation as a growth model.

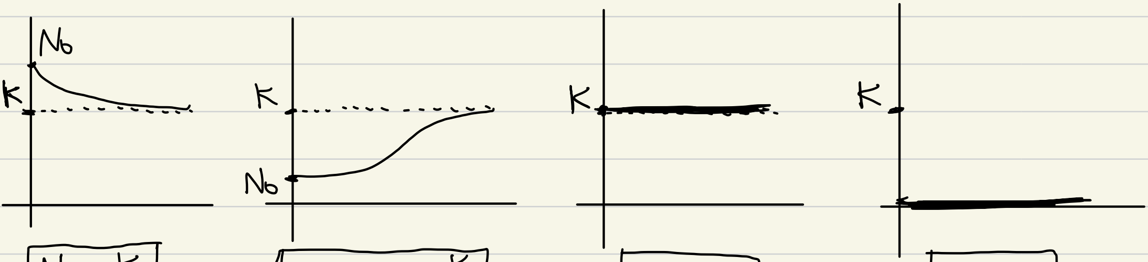
For the solution we found,

$$\lim_{t\to\infty} N(t) = K.$$

K is called the <u>carrying capacity</u> of the population.

$$N(t) = \frac{K}{1+\left(\frac{K}{N_0}-1\right)e^{-rt}}$$

Graphs of the solution $N(t)$ for different values of N_0.



$\boxed{N_0 > K}$ $\boxed{0 < N_0 < \frac{K}{2}}$ $\boxed{N_0 = K}$ $\boxed{N_0 = 0}$

We see that:

- when $N_0 > K$, the population decreases asymptotically towards K.
- when $N_0 < K$, the population increases towards the value K.
- when $N_0 = K$, the population remains constant and equal to K. (This corresponds to the constant solution $N(t) = K$ of the DE)
- when $N_0 = 0$, then the population remains equal to 0 (there is nothing to reproduce).

Solution of the logistic DE with $N_0 = 0$.

$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$. The function $N(t)=0$ is a constant solution.

What does the logistic equation

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$$

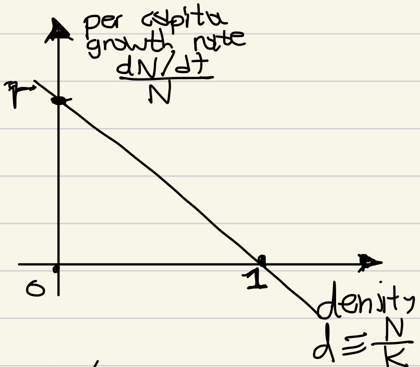
imply about the population growth?

According to it, the per capita growth rate

$$\frac{dN/dt}{N} = r\left(1-\frac{N}{K}\right)$$

is not constant, but depends on the "density" $\frac{N}{K}$. To be more specific,

$\frac{dN/dt}{N}$ is proportional to $1-\frac{N}{K}$.



$$\frac{dN/dt}{N} = r\left(1-\frac{N}{K}\right)$$

The bigger the "density" is, the smaller the per capita growth rate gets.

* Compare and contrast with the exponential growth model, where the per capita growth rate is constant.

• EQUILIBRIA AND STABILITY

The DE for exp. growth $\frac{dN}{dt} = rN$,
the logistic eq. $\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$,

the von Bertalanffy DE are all examples of autonomous DE's:

$$\boxed{\frac{dy}{dt} = g(y)} \qquad (*)$$

Consider an autonomous DE of the form $(*)$. We say that the constant function

$$y(t) = \hat{y}$$

is a <u>point equilibrium</u> (or simply <u>equilibrium</u>) of the DE $(*)$ if

$$g(\hat{y}) = 0.$$

Example: Find the point equilibria of the DE

$$y' = 2(y+1)(y-2).$$

• This is a DE of the form $y' = g(y)$ with $g(y) = 2(y+1)(y-2)$.

In order to find the point equilibria of the DE we solve

$$g(y) = 0 \iff 2(y+1)(y-2) = 0$$
$$\iff y = -1 \quad \text{OR} \quad y = 2.$$

The equilibria are the two functions
$y_1(t) = -1$ and $y_2(t) = 2$.

<u>REMARK</u>: Here $\hat{y}$ is only a symbol. We could have said that the function

$$y(t) = c_0$$

is an equilibrium if and only if $g(c_0) = 0$.

Why are these functions called equilibria?
If we solve the DE $y' = g(y)$ with initial condition $y = \hat{y}$ (where $\hat{y}$ is an equilibrium), then we find the constant solution $y = \hat{y}$.

Indeed, at time 0 the quantity y is $\hat{y}$ and its growth rate is $\frac{dy}{dt} = g(\hat{y}) = 0$

Let us return to the logistic equation
$$\frac{dN}{d\tau} = rN\left(1 - \frac{N}{K}\right).$$

This is an autonomous DE $y' = g(y)$ with
$$g(y) = ry\left(1 - \frac{y}{K}\right).$$

Equilibria:
$$g(y) = 0 \iff y = 0 \quad \text{or} \quad y = K.$$

The point equilibria are $N=0$ and $N=K$. We have seen that if the initial population is $N_0 = 0$ or $N_0 = K$, then the population will remain constant.

Example: Find the equilibria of the DE
$$y' = (y^2+1)y.$$

- $g(y) = (y^2+1)y$

  $g(y) = 0 \iff y = 0.$

  So the point equilibrium of the DE is
  $$y(t) = 0.$$

We are interested in the stability of point equilibria of the DE $y' = g(y)$.

Assume $\hat{y}$ is an equilibrium of $y' = g(y)$.

- If $g'(\hat{y}) < 0$, then the equilibrium $\hat{y}$ is called <u>locally stable</u>.

- If $g'(\hat{y}) > 0$, then the equilibrium $\hat{y}$ is called <u>unstable</u>.

If $y(0) = \hat{y}$, the solution of the DE will be $y(t) = \hat{y}$.
What happens to the solution of the DE if we change the initial condition to $y(0) = \hat{y} + \delta$ (where δ is a small number)?

- When the equilibrium $\hat{y}$ is locally stable, the new solution $y(t)$ will satisfy
$$\lim_{t \to \infty} y(t) = \hat{y}.$$

- When the equilibrium $\hat{y}$ is unstable, the new solution will not tend towards $\hat{y}$.

Examples: (i) Consider the DE $\frac{dy}{dt} = y(y+1)$.
What are the equilibria of this DE?
Classify them according to stability.

- This is a DE $\frac{dy}{dt} = g(y)$, with

  $g(y) = y(y+1)$.

  We find the equilibria:

  $$g(y) = 0 \iff y(y+1) = 0$$
  $$\iff y = 0 \quad \text{OR} \quad y = -1.$$

  Equilibria: $y_1(t) = 0$ and $y_2(t) = -1$.

  $g'(y) = (y^2 + y)' = 2y + 1$.

  $g'(0) = 1 > 0$, so $y_1 = 0$ is an unstable point eq.

  $g'(-1) = -1 < 0$, so $y_2 = -1$ is a locally stable point equilibrium.

(ii) Find the stability of the equilibria of the logistic equation

$$\frac{dN}{d\tau} = rN\left(1 - \frac{N}{K}\right).$$

Explain what stability means for the population $N(t)$.

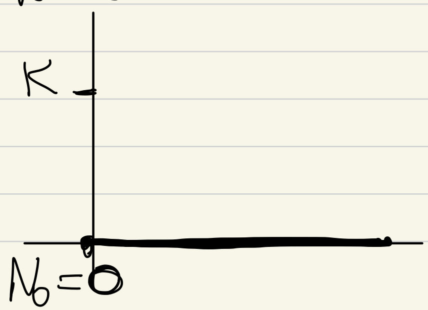
- $\frac{dN}{d\tau} = g(N)$ where $g(N) = rN\left(1 - \frac{N}{K}\right)$.

  $g(N) = 0 \Leftrightarrow N = 0$ OR $N = K$.

  $$g'(N) = r\left(1 - \frac{N}{K}\right) - \frac{rN}{K}$$
  $$= r - \frac{2rN}{K}.$$

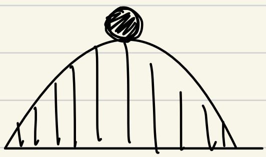
$g'(0) = r > 0$: $N = 0$ is an unstable eq.
$g'(K) = -r < 0$: $N = K$ is a locally stable eq.

When we perturb the initial condition $N_0 = 0$ by a small quantity, the population $N(t)$ will not evolve towards the direction of returning to 0.

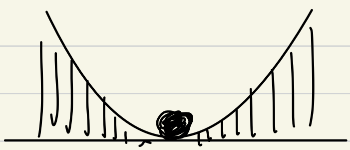


On the contrary, if we perturb the initial condition $N_0 = K$, the population will try to return to K.

A way to think of locally stable and unstable equilibrium points:



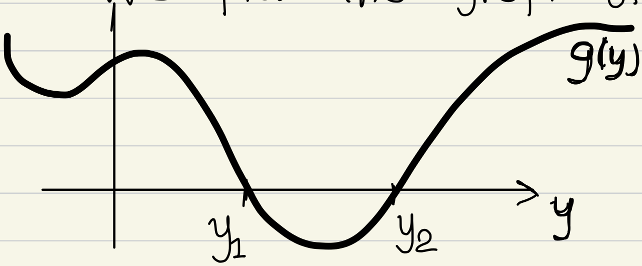
unstable equilibrium



locally stable equilibrium

The stability of equilibrium points can also be found graphically.
For the autonomous DE $\frac{dy}{dt} = g(y)$;

- We plot the graph of $g(y)$



- The equilibria of the DE are the points of intersection of the graph of $g(y)$ with the horizontal axis (solutions of $g(y)=0$)

In the example above, they are $y=y_1$ & $y=y_2$.

- If the function $g(y)$ is increasing near $\hat{y}$, then $\hat{y}$ is unstable.
If $g(y)$ is decreasing near $\hat{y}$, then $\hat{y}$ is locally stable.

In the example, y_1 is locally stable while y_2 is unstable.

(see Example 3, page 412).

- ## THE ALLEE EFFECT

Some populations have obstacles in reproduction when their numbers are small, because of lack of suitable mates.

A growth model that takes this fact into account:

$$\begin{cases} \dfrac{dN}{dT} = rN(N-a)\left(1 - \dfrac{N}{K}\right) \\ N(0) = N_0 \end{cases}$$

$a, K, r > 0$: positive constants
$0 < a < K$ by assumption

K : the carrying capacity
a : this constant is the treshold value below which the growth rate becomes negative.