Øv. 2, 8.1.34: Solve the D.E.

$$\frac{dy}{dx} = (3-y)(2+y).$$

(This is a separable DE, i.e. it has the form $y' = g(y)$.)

We have to divide by $(3-y)(2+y)$.

$(3-y)(2+y) = 0 \Leftrightarrow y = 3$ or $y = -2$.

So the constant functions
$y_1(x) = 3$ and $y_2(x) = -2$
are solutions of the DE.

When $(3-y)(2+y) \neq 0$,

$$\frac{dy}{dx} = (3-y)(2+y) \Rightarrow \frac{dy}{(3-y)(2+y)} = dx$$

$$\Rightarrow \int \frac{dy}{(3-y)(2+y)} = \int dx. \quad (*)$$

We have to calculate $\int \frac{dy}{(3-y)(2+y)}$.

We look for coefficients $A, B \in \mathbb{R}$ such that

$$\frac{1}{(3-y)(2+y)} = \frac{A}{3-y} + \frac{B}{2+y} \Rightarrow$$

$$A(2+y) + B(3-y) = 1 \Rightarrow$$

$$(A-B)y + (2A+3B) = 1, \text{ for all } y \in \mathbb{R}.$$

So

$$\begin{array}{l|l} A - B = 0 & A = \frac{1}{5} \\ 2A+3B = 1 & B = \frac{1}{5} \end{array}$$

Thus

$$\frac{1}{(3-y)(2+y)} = \frac{1}{5}\left(\frac{1}{3-y} + \frac{1}{2+y}\right).$$

$$(*) \Rightarrow \frac{1}{5}\int \frac{dy}{3-y} + \frac{1}{5}\int \frac{dy}{2+y} = \int dx$$

$$\Rightarrow \frac{1}{5}\ln|2+y| - \frac{1}{5}\ln|3-y| = x + c$$

$$\Rightarrow \frac{1}{5}\ln\left|\frac{2+y}{3-y}\right| = x + c$$

$$\Rightarrow \quad \frac{1}{5} \ln\left|\frac{2+y}{3-y}\right| = x + c$$

$$\Rightarrow \quad \ln\left|\frac{2+y}{3-y}\right| = 5x + c \,, \quad c \in \mathbb{R} \text{ constant}$$

$$\Rightarrow \left|\frac{2+y}{3-y}\right| = e^{5x+c} = e^{c} \cdot e^{5x}$$

$$= C_1 e^{5x} \,, \quad C_1 > 0 \quad \text{constant}$$

$$\Rightarrow \frac{2+y}{3-y} = \pm C_1 e^{5x} \,, \quad C > 0 \quad \text{constant}$$

$$\Rightarrow \frac{2+y}{3-y} = K e^{5x} \,, \quad K \neq 0 \quad \text{constant}$$

Solving for y we get

$$2 + y = 3Ke^{5x} - Ke^{5x} y \quad \Rightarrow$$

$$(1 + Ke^{5x}) y = 3Ke^{5x} - 2 \quad \Rightarrow$$

$$y(x) = \frac{3Ke^{5x} - 2}{Ke^{5x} + 1} \,, \quad K \neq 0 \text{ constant.}$$

Qu. 4, 8.2.5 : logistic equation
inner growth rate is $r = 1.5$
carrying capacity : $K = 100$

(a) Find the DE
(b) Find the equilibria and their stability.

The logistic equation is

$$\frac{dN(t)}{dt} = r\, N(t) \cdot \left(1 - \frac{N(t)}{K}\right)$$

where $r > 0$ is the intrinsic growth rate and $K > 0$ is the carrying capacity of the population.

Therefore

$$\frac{dN}{dt} = \frac{3}{2} N \left(1 - \frac{N}{100}\right).$$

We set

$$g(N) = \frac{3}{2} N \left(1 - \frac{N}{100}\right) = \frac{3N}{2} - \frac{3N^2}{200}.$$

We solve $g(N) = 0 \iff$

$$N = 0 \text{ or } N = 100.$$

The equilibria are $N=0$ and $N=100$.

$$g'(N) = \frac{3}{2} - \frac{3N}{100}$$

$g'(0) = \frac{3}{2} > 0$, so $N=0$ is unstable.

$g'(100) = \frac{3}{2} - \frac{3 \cdot \cancel{100}}{\cancel{100}} = -\frac{3}{2} < 0$,

so $N = 100$ is locally stable.

(c) Solve the DE you wrote in (a) when the population at time 0 is equal to $N(0) = 60$.

$$\frac{dN}{dt} = \frac{3}{2} N\left(1 - \frac{N}{100}\right) \Rightarrow$$

$$\frac{dN}{dt} = \frac{3}{200} N(100 - N) \Rightarrow$$

$$\int \frac{dN}{N(100-N)} = \int \frac{3}{200}\, dt \Rightarrow \left(\begin{array}{c}\text{partial fraction}\\ \text{decompos.}\end{array}\right) \Rightarrow \left(\begin{array}{c}\text{find the}\\ \text{constant by}\\ N(0)=60\end{array}\right).$$

Øv. 7 (Exam 2019 s., opp. 2):

3 age groups: 0-, 1-, 2- year olds.
~~First year of life: no offspring.~~
3/4 survive the first year of life.
~~The 1-y.o. have on average 5 cubs.~~
One third survives the 2nd year of life.
~~The 2-y.o. have on average 2 cubs.~~

(a) Find the Leslie matrix L that describes this population?

(b) L has an eigenvalue with eigenv. $\begin{bmatrix} 16 \\ 6 \\ 1 \end{bmatrix}$. Find this eigenvalue.

(c) If we begin with 80 - 30 - 5 how many years does it take for the population to become 5 × its initial value?

d) Find the evolution of the population.

SOLUTION

a) L is going to be a 3×3 matrix because there exist 3 age groups.

$$L = \begin{bmatrix} 0 & 5 & 2 \\ 3/4 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix}.$$

* Conversely, if we were given L we can deduce the information given by the exercise.

(b) $$\begin{bmatrix} 0 & 5 & 2 \\ 3/4 & 0 & 0 \\ 0 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} 16 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} 32 \\ 12 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 16 \\ 6 \\ 1 \end{bmatrix}$$

so the corresponding eigenvalue is $\lambda = 2$.

(c) The population at time 0 is

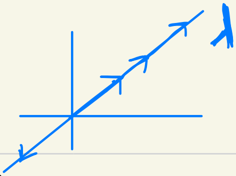
$$N(0) = \begin{bmatrix} 80 \\ 30 \\ 5 \end{bmatrix}.$$

We know that the population at time $t \geq 1$ is going to be

$$N(t) = L^t N(0).$$

We observe that

$$\begin{bmatrix} 80 \\ 30 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \cdot 16 \\ 5 \cdot 6 \\ 5 \cdot 1 \end{bmatrix} = 5 \begin{bmatrix} 16 \\ 6 \\ 1 \end{bmatrix},$$



so $N(0)$ is an eigenvector corresponding to the eigenvalue $\lambda = 2$.

* $N(0)$ is an eigenvector corresponding to $\lambda = 2$ because it is equal to

$$N(0) = 5\,\mathbf{v}, \quad \text{with} \quad \mathbf{v} = \begin{bmatrix} 16 \\ 6 \\ 1 \end{bmatrix}.$$

We have found that $\mathbf{v}$ is an eigenvector corresponding to $\lambda = 2$, and thus

$$\begin{aligned} L \cdot N(0) = L \cdot (5\mathbf{v}) &= 5 \cdot L\mathbf{v} \\ &= 5 \cdot (2\mathbf{v}) \\ &= 2 \cdot (5\mathbf{v}) \\ &= 2\,N(0). \end{aligned}$$

Therefore $L^t N(0) = \lambda^t N(0) = 2^t N(0)$. So in order for the population to be at least five times its initial value, we need

$$2^t \geq 5 \Rightarrow t \geq 3.$$

So at least 3 years have to pass.

(d) We know that if $\mathbf{v}$ is the eigenvector corresponding to the bigger eigenvalue λ of L, then:

- $\mathbf{v}$ is a stable age distribution (the percentages of individuals in each age group will converge to the ones in $\mathbf{v}$)
- λ is the growth parameter of the population.

Here we have to find all eigenvalues of L.

$$\det(\lambda I - L) = \begin{vmatrix} \lambda & -5 & -2 \\ -3/4 & \lambda & 0 \\ 0 & -1/3 & \lambda \end{vmatrix}$$

$$= \lambda \begin{vmatrix} \lambda & 0 \\ -1/3 & \lambda \end{vmatrix} + 5 \begin{vmatrix} -3/4 & 0 \\ 0 & \lambda \end{vmatrix} - 2 \begin{vmatrix} -3/4 & \lambda \\ 0 & -1/3 \end{vmatrix}$$

$$= \lambda(\lambda^2 - 0) - 5 \cdot \frac{3\lambda}{4} - 2 \cdot \frac{\cancel{3}}{4} \cdot \frac{1}{\cancel{3}}$$

$$= \lambda^3 - \frac{15\lambda}{4} - \frac{1}{2}$$

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To find all eigenvalues, we need to solve

$$\det(\lambda I - L) = 0 \iff \lambda^3 - \frac{15\lambda}{4} - \frac{1}{2} = 0.$$

We know that $\lambda = 2$ is a solution, so the polynomial $\lambda - 2$ divides $\lambda^3 - \frac{15\lambda}{4} - \frac{1}{2}$.

$$
\begin{array}{rrrr|l}
\lambda^3 & +\,0\lambda^2 & -\frac{15}{4}\lambda & -\frac{1}{2} & \lambda - 2 \\
\cline{5-5}
\lambda^3 & -\,2\lambda^2 & +\,0\lambda & +\,0 & \lambda^2 + 2\lambda + \frac{1}{4} \\
\hline
 & 2\lambda^2 & -\frac{15}{4}\lambda & -\frac{1}{2} & \\
 & 2\lambda^2 & -\,4\lambda & & \\
\cline{2-4}
 & & \frac{1}{4}\lambda & -\frac{1}{2} & \\
 & & \frac{1}{4}\lambda & -\frac{1}{2} & \\
\cline{3-4}
 & & & 0 & \\
\end{array}
$$

So $\lambda^3 - \frac{15\lambda}{4} - \frac{1}{2} = (\lambda - 2)\left(\lambda^2 + 2\lambda + \frac{1}{4}\right)$

and

$$\lambda^3 - \frac{15\lambda}{4} - \frac{1}{2} = 0 \iff$$

$$(\lambda-2)\left(\lambda^2 + 2\lambda + \frac{1}{4}\right) = 0 \iff$$

$$\lambda = 2 \quad \text{or} \quad \lambda^2 + 2\lambda + \frac{1}{4} = 0$$

$$\lambda = -1 - \frac{\sqrt{3}}{2} \quad \text{or} \quad \lambda = -1 + \frac{\sqrt{3}}{2}.$$

So the bigger eigenvalue is $\lambda = 2$ and the corresponding eigenvector

$$V = \begin{bmatrix} 16 \\ 6 \\ 1 \end{bmatrix}$$

is a stable age distribution.

So in the long run, the population will consist of approximately

$\frac{16}{23} \cong 69{,}57\,\%$ 0-year olds,

$\frac{6}{23} \cong 26.07\,\%$ 1-year olds, and

$\frac{1}{6} \approx 16{,}66\ \%$ 2-year olds.

Øv. 5, 9.2.21: Let $A = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$.

(a) $AB = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 0 & 5 \end{bmatrix}$.

(b) $BA = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 6 \\ 2 & 2 \end{bmatrix}$.

(So $AB \neq BA$).

Øv. 5, 9.2.15: Find the transpose of $A = \begin{bmatrix} -1 & 0 & 3 \\ 2 & 1 & -4 \end{bmatrix}$.

$$A^T = \begin{bmatrix} -1 & 2 \\ 0 & 1 \\ 3 & -4 \end{bmatrix}.$$

Ov. 5 9.1.6 : Find the solutions of

$$\begin{cases} -2x + 3y = 5 \\ ax - y = y \end{cases}$$

with respect to a.

For which values of a do there exist zero sol's, a unique sol., inf. many sol's?

- $$\begin{cases} -2x + 3y = 5 \\ ax - 2y = 0 \end{cases} \quad \Bigg| \quad \begin{aligned} &-2x + \frac{3a}{2}x = 5 \\ &y = \frac{a}{2}x \end{aligned}$$

$$\begin{cases} \left(\frac{3a}{2} - 2\right)x = 5 \\ y = \frac{a}{2}x. \end{cases}$$

We have $\frac{3a}{2} - 2 \neq 0 \iff a \neq \frac{4}{3}$.

- When $a \neq \frac{4}{3}$, we have

$$x = \frac{5}{\frac{3a}{2} - 2} = \frac{10}{3a-4}$$

and

$$y = \frac{a}{2} \cdot \frac{10}{3a-4} = \frac{10a}{6a-8}.$$

The system has a unique solution.

- When $a = \frac{4}{3}$, the system is written as $\begin{cases} 0 = 5 \\ y = \frac{2x}{3} \end{cases}$ which has no solutions.