$$\begin{bmatrix} 4 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Exercise: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$

(i) Find eigenvalues & eigenvectors of A

(ii) Find $A^{10}\mathbf{v}$, where $\mathbf{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

(i) We have found that A has eigenvalues $\lambda_1 = -1$, $\lambda_2 = 4$ with eigenvectors

$$x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \text{ respectively.}$$

(ii) We analyse $\mathbf{v}$ as

$$\mathbf{v} = \alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2.$$

(This is called a linear combination of the vectors $\mathbf{x}_1$ and $\mathbf{x}_2$).

$$\alpha_1 \mathbf{x}_1 + \alpha_2 \mathbf{x}_2 = \mathbf{v} \Rightarrow \alpha_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \frac{1}{3+2} \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

So $\alpha_1 = 2$ and $\alpha_2 = 1$.
Therefore

$$v = 2x_1 + x_2.$$

We now find $A^{10}v$.

$$A^{10}v = A^{10}(2x_1 + x_2)$$
$$= 2 \cdot A^{10}x_1 + A^{10}x_2$$
$$= 2 \cdot \lambda_1^{10}x_1 + \lambda_2^{10}x_2$$
$$= 2 \cdot (-1)^{10}x_1 + 4^{10}x_2$$
$$= 4^{10}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
$$= \begin{bmatrix} 2 \cdot 4^{10} + 1 \\ 3 \cdot 4^{10} - 2 \end{bmatrix}.$$

Exercise: Find the eigenvalues und eigenvectors of

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 3 & -1 \end{bmatrix}.$$

- $$\det(\lambda I - A) = \begin{vmatrix} \lambda+1 & -1 & 0 \\ -1 & \lambda-2 & -1 \\ 0 & -3 & \lambda+1 \end{vmatrix}$$

$$= (\lambda+1)\begin{vmatrix} \lambda-2 & -1 \\ -3 & \lambda+1 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 0 & \lambda+1 \end{vmatrix}$$

$$= (\lambda+1)\left[(\lambda-2)(\lambda+1) - 3\right] - (\lambda+1)$$

$$= (\lambda+1)(\lambda^2 - \lambda - 2 - 3 - 1)$$

$$= (\lambda+1)(\lambda^2 - \lambda - 6)$$

$$= (\lambda+1)(\lambda-3)(\lambda+2).$$

So $\det(\lambda I - A) = 0 \Rightarrow \lambda = -1$ or $\lambda = 3$ or $\lambda = -2$.

Eigenvalues of A: $\lambda_1 = -1$, $\lambda_2 = 3$, $\lambda_3 = -2$.

Eigenvector for $\lambda_1 = -1$:

$$Ax = -x \Rightarrow \begin{bmatrix} 0 & -1 & 0 \\ -1 & -3 & -1 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_2 = 0 \\ x_1 + x_3 = 0 \end{cases}$$

So $x = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ is an eigenvector.

Eigenvector for $\lambda_2 = 3$:

$$Ay = 3y \Rightarrow \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We use Gauss elimination.

$$\left[\begin{array}{ccc:c} 4 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -3 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} -1 & 1 & -1 & 0 \\ 4 & -1 & 0 & 0 \\ 0 & -3 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc:c} -1 & 1 & -1 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & -3 & 4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:c} 1 & -1 & 1 & 0 \\ 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

One eigenvector is $y = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$.

Eigenvector for $\lambda_3 = -2$:

$$Az = -2z \Rightarrow \begin{bmatrix} -1 & -1 & 0 \\ -1 & -4 & -1 \\ 0 & -3 & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $z_2 = 1$ and $z_3 = -3$, we get $z_1 = -1$.
So

$$z = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$ is an eigenvector.

- LESLIE MATRICES REVISITED: AN APPLICATION OF EIGENVECTORS AND EIGENVALUES

Recall that if $L = \begin{bmatrix} F_0 & F_1 & \cdots & F_{m-1} \\ P_0 & 0 & \cdots & 0 \\ 0 & P_1 & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 0 \end{bmatrix}$

is the Leslie matrix of a population, then:

- the females are split into m age groups
- F_i is the offspring-rate of the i-th age group $(i=0,\dots,m-1)$
- P_i is the survival rate of the i-th age group $(i=0,\dots,m-1)$.

Suppose the Leslie matrix is

$$L = \begin{bmatrix} 1.5 & 2 \\ 0.08 & 0 \end{bmatrix}$$

and we start with the population vector

$$N(0) = \begin{bmatrix} 100 \\ 100 \end{bmatrix}.$$

We can use the relation

$$N(t+1) = L\,N(t)$$

to find successive population vectors. These are:

$$\begin{bmatrix}100\\100\end{bmatrix}, \begin{bmatrix}350\\8\end{bmatrix}, \begin{bmatrix}541\\28\end{bmatrix}, \begin{bmatrix}868\\43\end{bmatrix}, \begin{bmatrix}1388\\69\end{bmatrix}, \begin{bmatrix}2221\\111\end{bmatrix}, \ldots$$

Let us look at the successive ratios of 0-year olds, i.e.

$$q_0(t) = \frac{N_0(t)}{N_0(t-1)}, \quad t = 1, 2, \ldots$$

These are:

3.5, 1.55, 1.60, 1.5991, 1.6001, 1.5997, 1.6001...

It seems like these ratios converge to a specific number $(=1.6)$ when $t \to \infty$.

What about the corresponding quotients for the 1-year-olds? We have the numbers

0.08, 3.5, 1.536, 1.605, 1.609, 1.604, ...

and it seems that these quotients converge to the same number.

We now look at the proportion of the 0-year olds in the total population, namely

$$P_0(t) = \frac{N_0(t)}{N_0(t)+N_1(t)}, \quad t=0,1,2,\dots$$

Its values are:

$0.5, \quad 0.9777, \quad 0.9528, \quad 0.9526, \quad \dots$

and again it looks like these percentages converge to $\cong 95.2\%$.

This means that asymptotically, the population will tend to consist of

$\cong 95.2\%$ 0-y.o. and

$\cong 4.8\%$ 1-y.o

In other words, these observations suggest that the population vectors tend to become multiples of one vector $\mathbf{v}$ for which

$$L\mathbf{v} = \lambda \mathbf{v} \quad (\text{where } \lambda = 1.6).$$

We know that $\mathbf{v}$ is an eigenvector for L. This vector is called a <u>stable age distribution</u> vector.

How do we find the stable age distribution in general?

- For $L = \begin{bmatrix} 1.5 & 2 \\ 0.08 & 0 \end{bmatrix}$ as in our example we find eigenvalues and eigenvectors.

$$\det(\lambda I - L) = \begin{vmatrix} \lambda - 1.5 & -2 \\ -0.08 & \lambda \end{vmatrix}$$

$$= \lambda(\lambda - 1.5) - 0.16$$

$$= \lambda^2 - 1.5\lambda - 0.16$$

We solve $\det(\lambda I - L) = 0 \iff$
$$\lambda^2 - 1.5\lambda - 0.16 = 0.$$

$$D = (-1.5)^2 + 4 \cdot 0.16 = \frac{289}{100}$$

$$\lambda_{1,2} = \frac{1}{2}\left(\frac{3}{2} \pm \frac{17}{10}\right) \quad \begin{cases} \lambda_1 = \frac{8}{5} = 1.6 \\ \lambda_2 = -\frac{1}{10} = -0.1 \end{cases}$$

We now find the corresponding eigenvectors.

- For $\lambda_1 = 1.6$:

$$Lx = \frac{8}{5}x \Longrightarrow \begin{bmatrix} 0.1 & -2 \\ -0.08 & 1.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow 0.1\,x_1 - 2x_2 = 0$$
$$\Longrightarrow x_1 - 20x_2 = 0$$

$x = \begin{bmatrix} 20 \\ 1 \end{bmatrix}$ is an eigenvector.

- For $\lambda_2 = -0.1$:

$$Ly = -\frac{1}{10}y \Longrightarrow \begin{bmatrix} -1.6 & -2 \\ -0.08 & -0.1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow 1.6\,y_1 + 2y_2 = 0$$

$$\Longrightarrow 4y_1 + 5y_2 = 0.$$

$y = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$ is an eigenvector.

Let's suppose we start studying the population when $N(0) = \begin{bmatrix} 105 \\ 1 \end{bmatrix}$.

We express this vector as a "linear combination" of the eigenvectors x and y of L.

$$N(0) = \begin{bmatrix} 105 \\ 1 \end{bmatrix} = 5 \cdot \begin{bmatrix} 20 \\ 1 \end{bmatrix} + \begin{bmatrix} 5 \\ -4 \end{bmatrix} = 5\mathbf{x} + \mathbf{y}.$$

The population vector $N(t)$ at time t is

$$\begin{aligned} N(t) &= L^t \cdot N(0) \\ &= L^t (5\mathbf{x} + \mathbf{y}) \\ &= 5 L^t \mathbf{x} + L^t \mathbf{y} \\ &= 5(1.6)^t \mathbf{x} + (-1)^t \mathbf{y}. \end{aligned}$$

Because the eigenvalue $\lambda_1 = 8/5$ is bigger, the first term will "dominate" in the sum and will describe the asymptotic behaviour of $N(t)$.

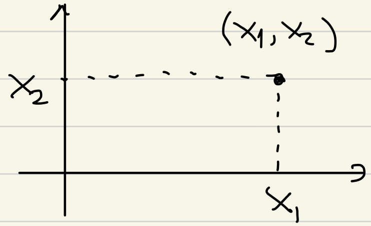
Suppose L is a 2×2 Leslie matrix with eigenvalues λ_1 and λ_2.

- The larger eigenvalue is the growth parameter of the population.
- The eigenvector that corresponds to the larger eigenvalue is a stable age distribution.

(The same applies for Leslie matrices of higher dimensions, for example 3×3 matrices).

4. ANALYTIC GEOMETRY

We have already seen that the 2-dimensional plane can be identified with the set of ordered pairs (x_1, x_2), $x_1, x_2 \in \mathbb{R}$.



Generally, for any $n \geq 1$ we define

$$\mathbb{R}^n = \{ (x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbb{R} \}$$

Elements of $\mathbb{R}^n$ are represented as n-tuples $(x_1, x_2, \ldots, x_n)$ or -preferably- as n-dimensional column vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [x_1 \ x_2 \ \ldots \ x_n]^T$$

The numbers $x_1, x_2, \ldots, x_n \in \mathbb{R}$ are called the <u>components</u> or <u>coordinates</u> of $\mathbf{x}$.

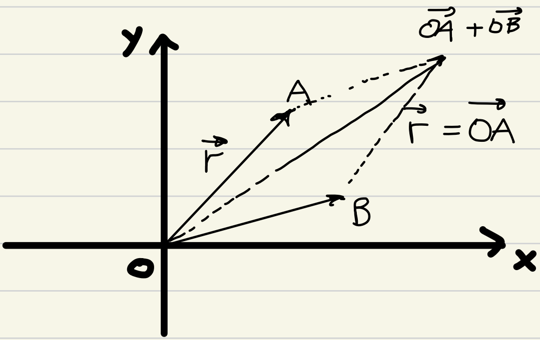
We have defined vector addition:

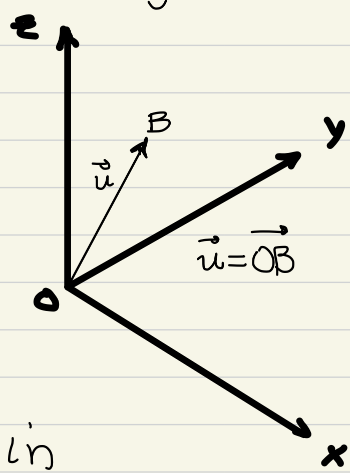
$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}$$

and multiplication of a vector by a scalar:

$$\lambda \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{bmatrix}.$$

For 2-dimensional & 3-dimensional vectors we may represent vector operations geometrically.





This is no longer possible in dimensions greater than 3. However the properties of the operations remain the same.