We solved inequalities of the form

$$|x| \leqslant \theta \quad , \text{ where } \theta \geqslant 0 .$$

What if $\theta < 0$?
In that case, inequality

$$|x| \leqslant \theta$$

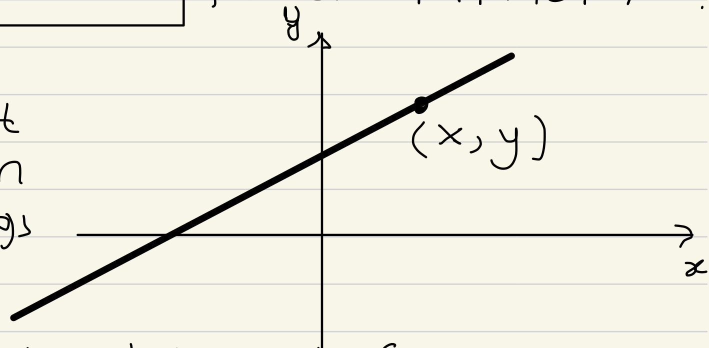
has NO SOLUTIONS because the absolute value of x is always non-negative: $|x| \geqslant 0$.

E.g. $|x| \leqslant -2$ has no solutions.

ANALYTIC GEOMETRY

Every line in the plane has an equation of the generic form

$$\boxed{Ax + By + C = 0}, \text{ where } |A| + |B| \neq 0.$$



This means that a point (x, y) in the plane belongs to the line if and only if the coordinates x and y satisfy

$$Ax + By + C = 0.$$

Example: The equation $2x - 3y + 1 = 0$ represents a line in the plane.

- The point $(4, 3)$ belongs to the line, because $2 \cdot 4 - 3 \cdot 3 + 1 = 8 - 9 + 1 = 0$.
- The point $(2, 2)$ does not, because $2 \cdot 2 - 3 \cdot 2 + 1 = -1 \neq 0$.

(*) The condition $|A| + |B| \neq 0$ simply means that the coefficients A and B cannot be 0 at the same time.

Some special cases:

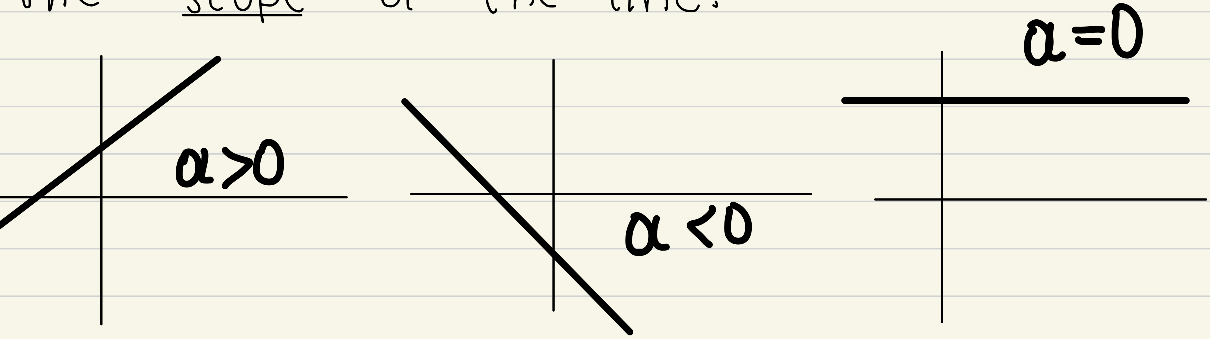
- When $B \neq 0$, the equation becomes

$$y = -\frac{A}{B}x - \frac{C}{B}$$

which can be written generically as

$$\boxed{y = ax + b}, \quad a, b \in \mathbb{R}.$$

The number $a \in \mathbb{R}$ is called the <u>slope</u> of the line.



Example: • The slope of the line

$$y = 2x - \frac{1}{2}$$

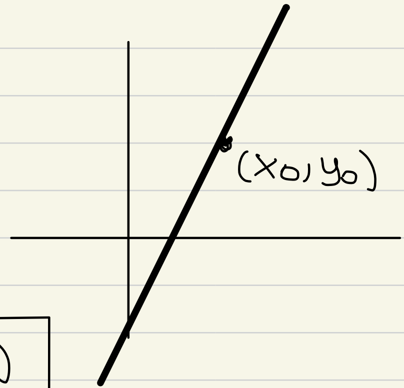
is equal to 2.

• What is the slope of $2x + y + 1 = 0$?

$2x + y + 1 = 0 \Leftrightarrow y = -2x - 1$

So the slope is equal to -2.

Suppose we have a given point (x_0, y_0) in the plane and we need to find the equation of the line that goes through this point and has slope equal to m.



This is

$$\boxed{y - y_0 = m \cdot (x - x_0)}$$

Example: Find the line that goes through the point $(-1, 2)$ and has slope equal to -2.

Here $x_0 = -1$, $y_0 = 2$ and $m = -2$.
The equation is

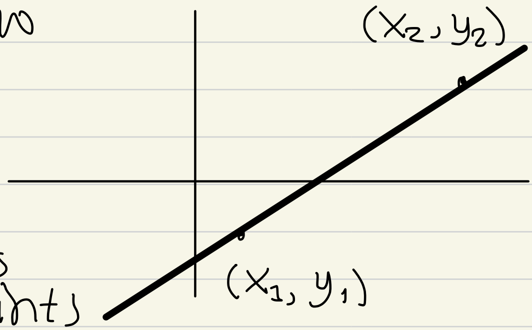
$$y - 2 = -2(x + 1) \Rightarrow$$

$$y - 2 = -2x - 2 \Rightarrow$$

$$y = -2x.$$

$$y = ax + b$$
$$y = cx + d$$
$$y = Dx + E$$

Suppose we are given two points (x_1, y_1), (x_2, y_2) in the plane.



The slope of the unique line that goes through these two points is equal to

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The equation of this unique line is

$$y - y_1 = m \cdot (x - x_1).$$

Example: Find the equation of the line that goes through the points $(1, 1)$ and $(2, 4)$.

- The slope of the line is
$$m = \frac{4-1}{2-1} = 3.$$

The equation is
$$y - 1 = 3(x-1) \Rightarrow y = 3x - 2.$$

Alternatively: $y - 4 = 3(x-2) \Rightarrow$
$$y = 3x - 2.$$

▶ Back to the generic equation

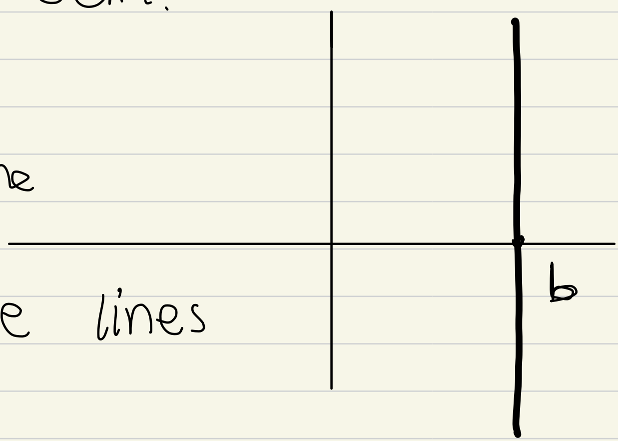
$$Ax + By + C = 0.$$

We studied all kinds of lines for which $B \neq 0$, but it might be that $B = 0$.

In that case (when $B=0$) we have the equation $Ax + C = 0$ with $A \neq 0$ and we get

$$\boxed{x = b}, \quad b \in \mathbb{R}.$$

The lines of the form $x=b$ are perpendicular to the horizontal axis.

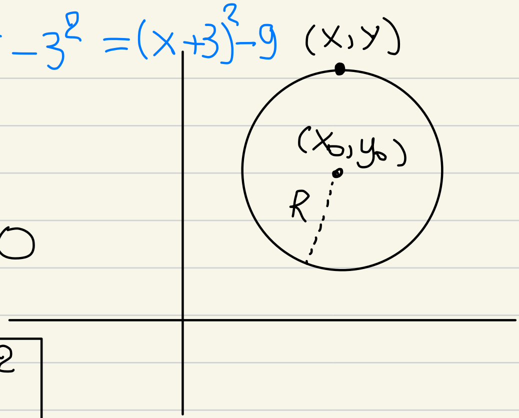


The slope of these lines is <u>not defined</u>.

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$x^2 + 6x = x^2 + 2\cdot 3 x + 3^2 - 3^2 = (x+3)^2 - 9$$

The equation of the circle in the plane with center equal to (x_0, y_0) and radius $R > 0$ is



$$\boxed{(x - x_0)^2 + (y - y_0)^2 = R^2}$$

Example: What is the equation of the circle with center $(-1, 1)$ and radius 2?

$$(x+1)^2 + (y-1)^2 = 4.$$

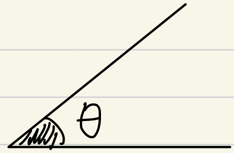
Example: What does the equation
$$x^2 - 2x + y^2 + 4y + 1 = 0$$
represent in the plane?

$$\underline{x^2 - 2x} + \underline{y^2 + 4y} + \underline{1} = 0 \Rightarrow$$
$$x^2 - 2x + 1 - 1 + y^2 + 4y + 4 - 4 + 1 = 0 \Rightarrow$$
$$(x^2 - 2x + 1) + (y^2 + 4y + 4) - 4 = 0 \Rightarrow$$
$$(x-1)^2 + (y+2)^2 = 2^2$$

Circle with center $(1, -2)$ and radius 2.

· TRIGONOMETRY

We measure angles in radiants instead of degrees.

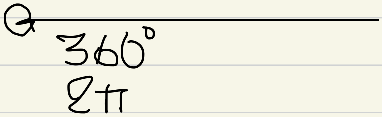


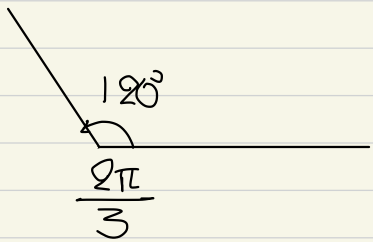
A full angle (360°) is equal to 2π rads. We use this to convert any angle in degrees to rads and vice versa.

E.g. How many radiants are 120°?

- 360° degrees are 2π rads.
  120° degrees are x rads.

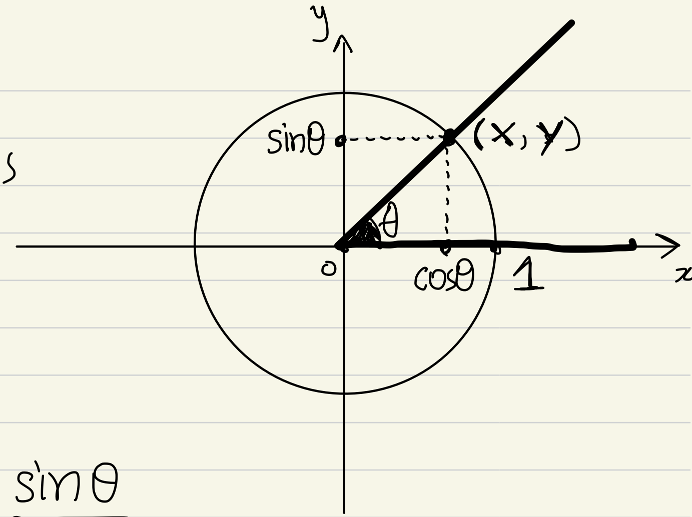
$$\frac{360}{120} = \frac{2\pi}{x} \Rightarrow x = \frac{120}{360} 2\pi = \frac{2\pi}{3}.$$





DEGREES	RADS
0°	0
30°	$\pi/6$
45°	$\pi/4$
60°	$\pi/3$
90°	$\pi/2$
180°	π

Definition of the trigonometric numbers of an angle θ.



$$\sin\theta = y$$

$$\cos\theta = x$$

$$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{x}{y} = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{1}{x} = \frac{1}{\cos\theta} \quad \text{(secant)}$$

$$\csc\theta = \frac{1}{y} = \frac{1}{\sin\theta} \quad \text{(cosecant)}$$

$$\boxed{\sin^2\theta + \cos^2\theta = 1}, \quad \theta \in \mathbb{R}.$$

$$\boxed{1 + \tan^2\theta = \frac{1}{\cos^2\theta}}$$

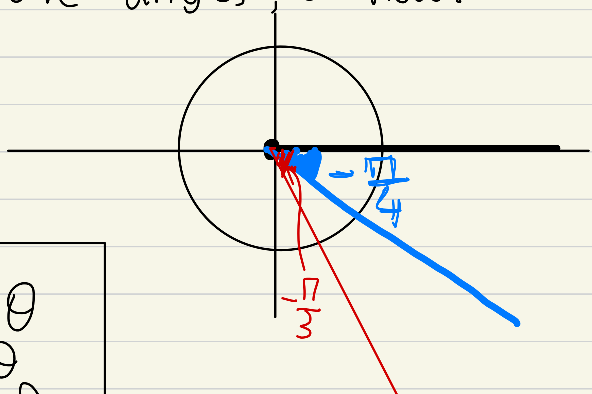
This identity follows directly from the basic identity. Indeed,

$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow$$

$$\frac{\sin^2\theta}{\cos^2\theta} + 1 = \frac{1}{\cos^2\theta} \Rightarrow$$

$$1 + \tan^2\theta = \frac{1}{\cos^2\theta}.$$

We can define negative angles, as well. We can extend the previous definitions of trigonometric numbers to negative angles.



$$\boxed{\begin{aligned} \sin(-\theta) &= -\sin\theta \\ \cos(-\theta) &= \cos\theta \\ \tan(-\theta) &= -\tan\theta \\ \cot(-\theta) &= -\cot\theta \end{aligned}}$$

Trigonometric Numbers of Given Angles.

θ	$\sin\theta$	$\cos\theta$
0	0 $\quad \frac{\sqrt{0}}{2}$	1
$\pi/6$	$\frac{1}{2}$ $\quad \frac{\sqrt{1}}{2}$	$\frac{\sqrt{3}}{2}$
$\pi/4$	$\frac{\sqrt{2}}{2}$ $\quad \frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\pi/3$	$\frac{\sqrt{3}}{2}$ $\quad \frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\pi/2$	1 $\quad \frac{\sqrt{4}}{2}$	0

Example: Solve the equation

$$2\sin\theta\cos\theta = \cos\theta, \quad 0 \leq \theta \leq 2\pi.$$

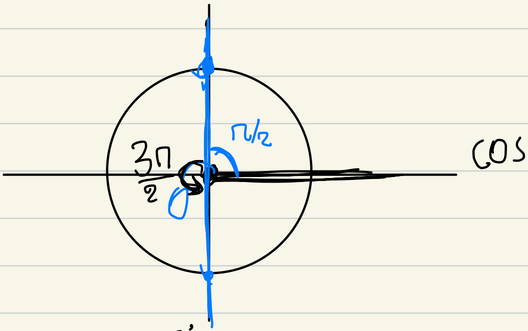
$$2\sin\theta\cos\theta = \cos\theta \Rightarrow$$

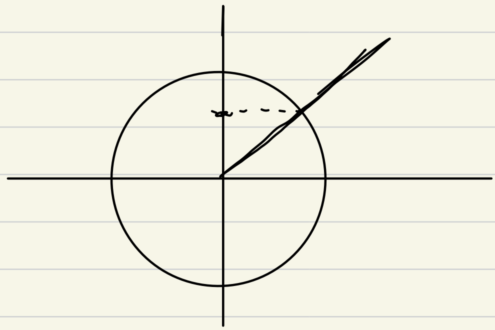
$$2\sin\theta\cos\theta - \cos\theta = 0 \Rightarrow$$

$$\cos\theta(2\sin\theta - 1) = 0 \Rightarrow$$

$$\cos\theta = 0 \quad \text{OR} \quad \sin\theta = \frac{1}{2}$$

We need to find all angles $0 \leq \theta \leq 2\pi$ such that either $\cos\theta = 0$ OR $\sin\theta = \frac{1}{2}$.





$$\cos\theta = 0 \implies \theta = \frac{\pi}{2} \quad \text{OR} \quad \theta = \frac{3\pi}{2}.$$