Exercise: Find the intervals of monotonicity as well as the local & global extrema of

$$f(x) = \frac{3}{2}x^4 - 2x^3 - 6x^2 + 2, \quad x \in \mathbb{R}.$$

- We find the derivative of f.

$$f'(x) = 6x^3 - 6x^2 - 12x$$

$$= 6x(x^2 - x - 2) \quad \leftarrow \text{abc-formula}$$

$$= 6x(x-2)(x+1)$$

x	$-\infty$		-1		0		2		$+\infty$
x		$-$		$-$	0	$+$		$+$	
$x-2$		$-$		$-$		$-$	0	$+$	
$x+1$		$-$	0	$+$		$+$		$+$	
$f'(x)$	$+\infty$	$-$	0	$+$	0	$-$	0	$+$	$+\infty$
$f(x)$		$\searrow$	$-\frac{1}{2}$	$\nearrow$	2	$\searrow$	-14	$\nearrow$	

$f \searrow (-\infty, -1]$, $f \nearrow [-1, 0]$, $f \searrow [0, 2]$, $f \nearrow [2, +\infty)$.

f has a loc. minimum at $x = -1$, the number $f(-1) = -\frac{1}{2}$

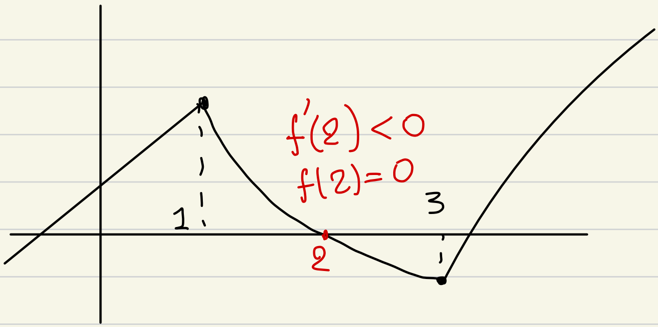
f has a loc. maximum at $x = 0$, the number $f(0) = 2$

f has a loc. minimum at $x = 2$, the number $f(2) = -14$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\frac{3}{2}x^4 - 2x^3 - 6x^2 + 2\right) = +\infty$$

$$\lim_{x\to+\infty} f(x) = +\infty$$

f does not have a global maximum.
f has a global minimum at $x = 2$, which is equal to $f(2) = -14$.



(NOT EXAMINABLE)

f is continuous at $[1,3]$
f is differentiable at $(1,3)$
(BUT NOT AT THE POINTS $x=1$, $x=3$)
$f'(x) < 0$ for all $x \in (1,3)$

$f \searrow [1,3]$

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

Exercise: Let $f:(0,+\infty) \to \mathbb{R}$, $f(x) = \dfrac{e^x}{x}$.

(i) Find the local and global extrema of f.

(ii) What is the range of f?

(i) $f(x) = \dfrac{e^x}{x}$, $x > 0$

$$f'(x) = \frac{(e^x)'x - e^x(x)'}{x^2} = \frac{e^x(x-1)}{x^2}, \quad x > 0.$$

$$f'(x) > 0 \iff \frac{e^x(x-1)}{x^2} > 0 \iff x - 1 > 0 \iff x > 1.$$

$$f'(x) < 0 \iff \frac{e^x(x-1)}{x^2} < 0 \iff x < 1.$$

x	0		1		$+\infty$
$f'(x)$	0	$-$	0	$+$	
$f(x)$		$\searrow$	e	$\nearrow$	

$f \searrow (0,1]$

$f \nearrow [1,+\infty)$.

f has a local min. at $x = 1$, which is equal to $f(1) = e$.

This is also a global minimum.

f has no global maximum.

(ii) $\displaystyle\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{e^x}{x}$

$$= \lim_{x\to+\infty} \frac{e^x}{1} = \lim_{x\to+\infty} e^x = +\infty.$$

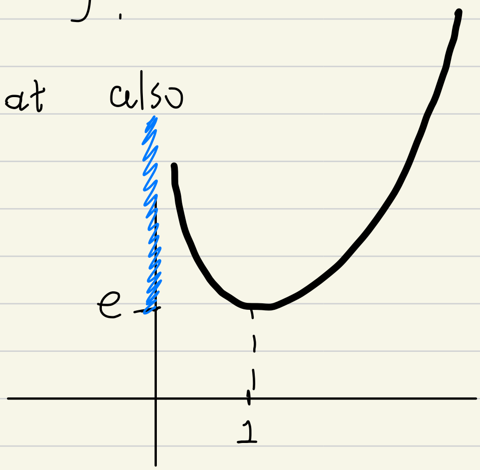
$[e, \infty)$

The range of f is equal to

$$f((0,+\infty)) = [e, +\infty).$$

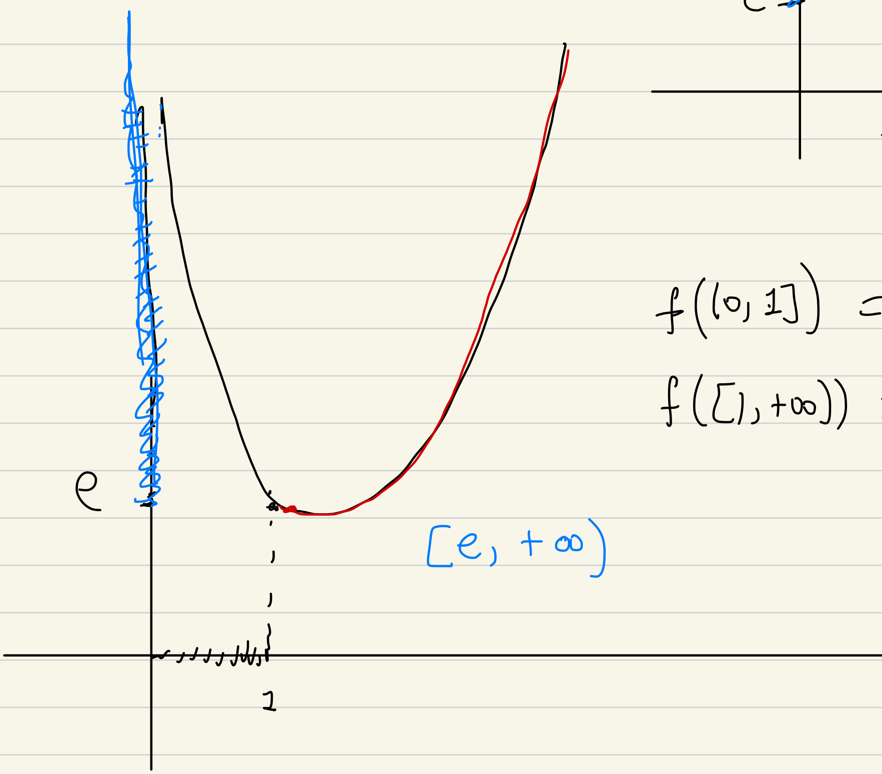
* We can actually see that also

$$\lim_{x \to 0^+} \frac{e^x}{x} = +\infty.$$

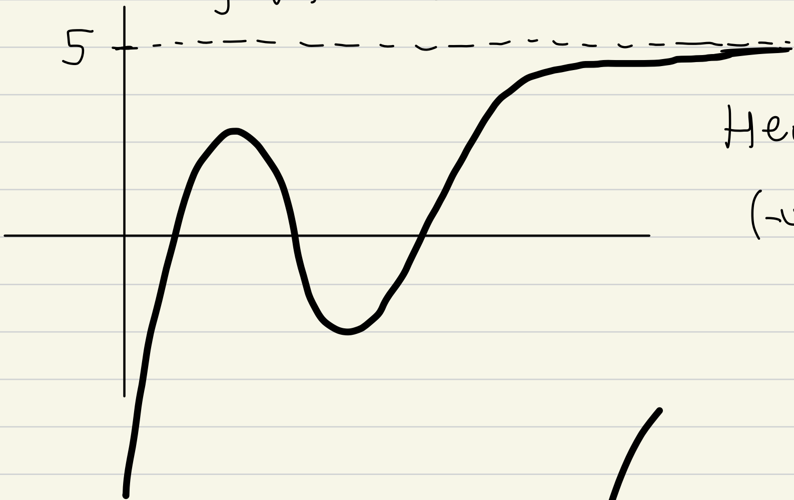


$f((0,1]) = [e, +\infty)$

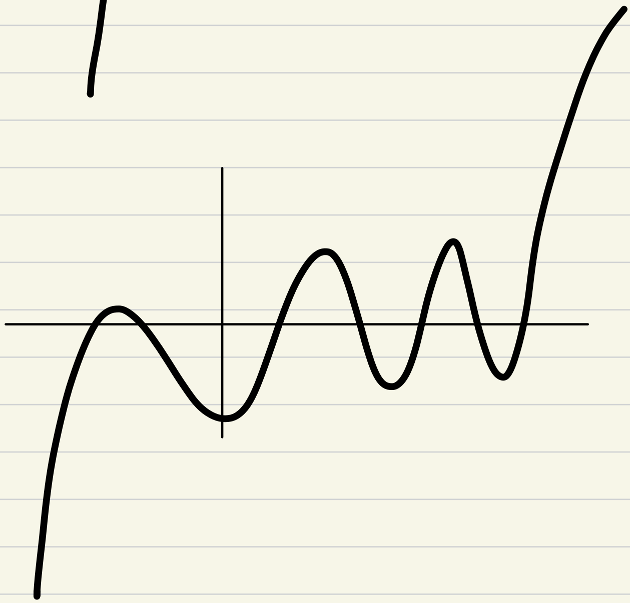
$f([1,+\infty)) = [e, +\infty)$



$f : (0, +\infty) \to \mathbb{R}$

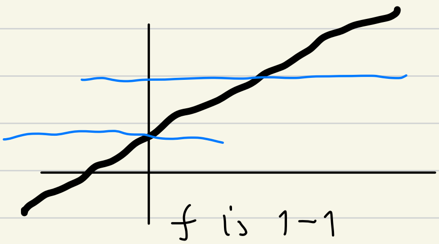


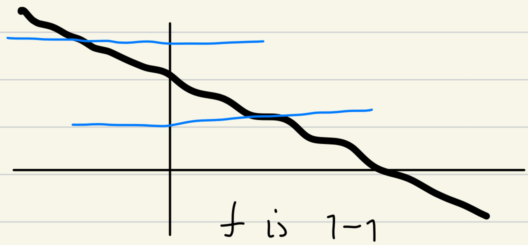
Here the range is $(-\infty, 5)$



Here it is $(-\infty, +\infty)$

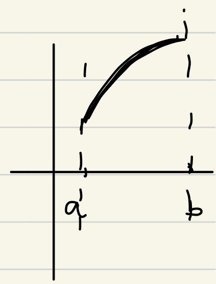
A very important fact regarding monotonicity: a strictly monotonic function (i.e. either strictly increasing or strictly decreasing) is always one-to-one (invertible).





As such, it can assume a real value $y \in \mathbb{R}$ <u>at most once</u> (i.e. once or it might not assume this value at all).

In particular, f can have at most one root in any interval where it is strictly monotonic.



! Suppose we are given an equation of the form $f(x)=0$. To show that this has precisely one root in $[a,b]$:
- We show it has at least one root in $[a,b]$.
- We show that f is monotonic on $[a,b]$.

$x^2 - x + 1 = 0$ $\qquad$ $\sin x + 3 \geq -1 + 3 = 2 > 0$

Exercise: How many roots does the equation

$$\sin x + 2x = 0$$

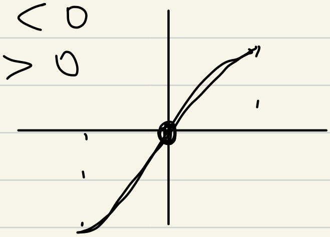
have on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$?

<u>ANSWER:</u> Set $f(x) = \sin x + 2x, \quad x \in [-\pi/2, \pi/2]$.

- f is continuous on $[-\pi/2, \pi/2]$
- $f(-\pi/2) = \sin(-\pi/2) - \pi = -1 - \pi < 0$

  $f(\pi/2) = \sin(\pi/2) + \pi = 1 + \pi > 0$

  thus $f(-\pi/2) \cdot f(\pi/2) < 0$



Therefore by the Intermediate Value theorem, there exists some $x_0 \in (-\pi/2, \pi/2)$ such that $f(x_0) = 0$.

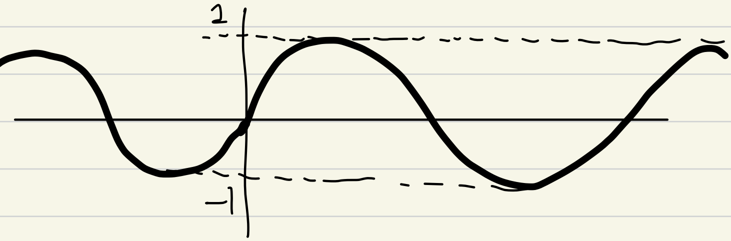
I.e. $f(x) = 0$ has at <u>least</u> one root in $[-\pi/2, \pi/2]$.

$$f'(x) = \underbrace{\cos x}_{\geq -1} + 2 \geq -1 + 2 = 1 > 0,$$

for all $x \in (-\pi/2, \pi/2)$, hence $f \nearrow [-\pi/2, \pi/2]$

Therefore the root of f in $[-\pi/2, \pi/2]$ must be unique.

The initial equation has precisely one root in $[-\pi/2, \pi/2]$.



$-1 \leq \sin x \leq 1$

$-1 \leq \cos x \leq 1$

Exercise: How many solutions does the equation

$$x^3 - 3x + 1 = 0$$

have on the interval $[-2, 2]$?
(HØST 2019, OPPGAVE 6)

- Set $f(x) = x^3 - 3x + 1$, $x \in [-2, 2]$.
(We need the number of roots of f)

$$f'(x) = 3x^2 - 3 = 3(x^2 - 1) = 3(x-1)(x+1)$$

x	-2		-1		1		2
$x-1$		$-$		$-$	0	$+$	
$x+1$		$-$	0	$+$		$+$	
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	-1	$\nearrow$	3	$\searrow$	-1	$\nearrow$	3

$f \nearrow [-2, -1]$
$f \searrow [-1, 1]$
$f \nearrow [1, 2]$

$$f(-2) = (-2)^3 - 3(-2) + 1 = -8 + 6 + 1 = -1$$
$$f(-1) = (-1)^3 - 3(-1) + 1 = 3$$
$$f(1) = 1 - 3 + 1 = -1$$
$$f(2) = 2^3 - 3 \cdot 2 + 1 = 3$$

On each of the intervals $[-2,-1]$, $[-1,1]$ and $[1,2]$, the function f has a root. This root is unique, because f is strictly monotonic there.

Therefore f has precisely 3 roots in $[-2,2]$.