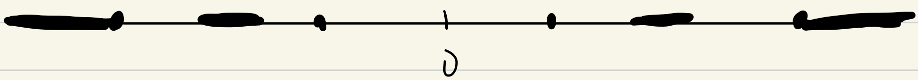
In order for $f: A \to \mathbb{R}$ to be either even or odd, A must satisfy the following:

$$x \in A \Rightarrow -x \in A.$$



If f is a given function, the domain of f is very often denoted by D_f.

Normally, the domain is given together with the function.
E.g. $f: [0,1] \to \mathbb{R}$, $f(x) = x^2 + 1$.

If not, we consider the domain of the function to be the biggest possible subset of $\mathbb{R}$ where $f(x)$ is well defined

E.g. • $f(x) = x^3 - 2x + 3$, $D_f = \mathbb{R}$

• $g(x) = 1 + \ln(x-1)$.

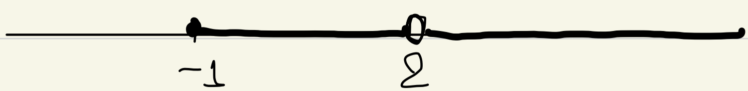
We need $x - 1 > 0 \iff x > 1$.
So $D_g = (1, +\infty)$.

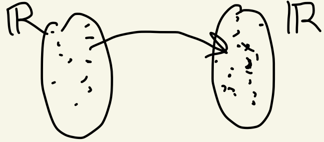
$\sqrt{a}$: well defined for $a \geqslant 0$

• $h(x) = \dfrac{1}{x-2} + \sqrt{x+1}$.

We need $x - 2 \neq 0 \iff x \neq 2$
and $x + 1 \geqslant 0 \iff x \geqslant -1$

So $D_h = [-1, 2) \cup (2, +\infty)$.





Exercise: Let $f(x) = 1 + e^{x-2}$
Find the (biggest possible) domain of f and the range of f.

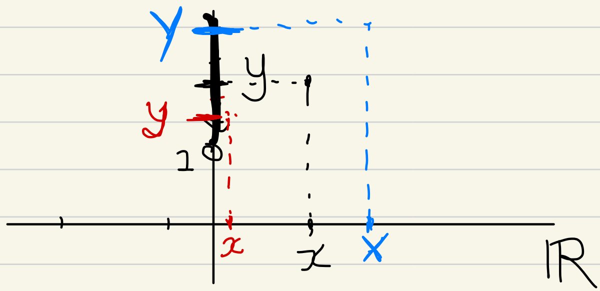
- $D_f = \mathbb{R}$.

We know that $e^x > 0$ for all $x \in \mathbb{R}$ therefore

$$f(x) = 1 + e^{x-2} > 1 \quad \text{for all } x \in \mathbb{R}.$$

We will show that

$$f(\mathbb{R}) = (1, +\infty).$$



Let $y \in (1, +\infty)$.
I need to show that there exists some value $x \in \mathbb{R}$ such that $f(x) = y$.
But

$$\begin{aligned} f(x) = y &\iff 1 + e^{x-2} = y \\ &\iff e^{x-2} = y - 1 \\ &\iff x - 2 = \ln(y-1) \\ &\iff x = 2 + \ln(y-1). \end{aligned}$$

For each $y > 1$ there exists $x \in \mathbb{R}$ such that $f(x) = y$. Therefore

$$f(\mathbb{R}) = (1, +\infty).$$

Suppose $f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$ are two functions. We say that f, g are <u>equal</u> and we write $f = g$, if:

- $A = B$
- $f(x) = g(x)$ for any $x \in A$.

For example, the functions

$$f: (0, +\infty) \to \mathbb{R}, \quad f(x) = \ln x - \ln 2 + 1$$
$$g: (0, \infty) \to \mathbb{R}, \quad g(x) = \ln\left(\frac{ex}{2}\right)$$

are equal. Indeed:

- they have the same domain $D_f = D_g = (0, \infty)$
- for any $x \in (0, \infty)$,

$$g(x) = \ln\left(\frac{ex}{2}\right) = \ln e + \ln x - \ln 2$$
$$= 1 + \ln x - \ln 2$$
$$= f(x).$$

The functions

$$f: \mathbb{R} \to \mathbb{R}, \quad f(x) = x^3 + 2$$
$$g: [0, 1] \to \mathbb{R}, \quad g(x) = x^3 + 2$$

are NOT equal. This is because $D_f \neq D_g$.

Exercise: Consider the functions

$f : \{-1, 0, 1\} \to \mathbb{R}, \quad f(x) = 2x,$

$g : \{-1, 0, 1\} \to \mathbb{R}, \quad g(x) = x^3 + x.$

Are f, g equal?

ANSWER

The domains of f, g are equal:

$$D_f = D_g = \{-1, 0, 1\}.$$

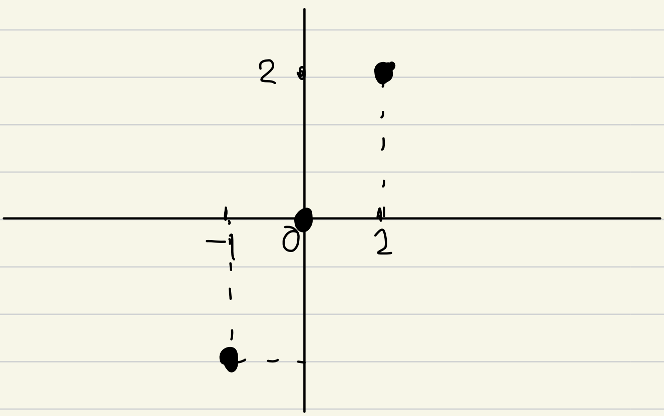
When $x = -1$: $f(-1) = -2$, $g(-1) = -2$

When $x = 0$: $f(0) = 0$, $g(0) = 0$

When $x = 1$: $f(1) = 2$, $g(1) = 2$

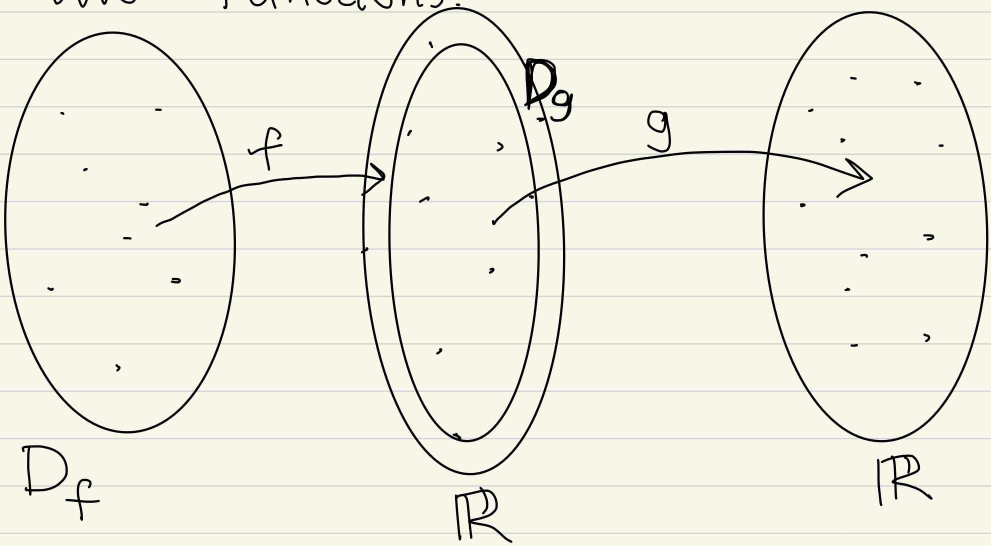
For any $x \in \{-1, 0, 1\}$, $f(x) = g(x)$.

So $f = g$.



COMPOSITION OF FUNCTIONS

Suppose $f : D_f \to \mathbb{R}$, $g : D_g \to \mathbb{R}$ are two functions.

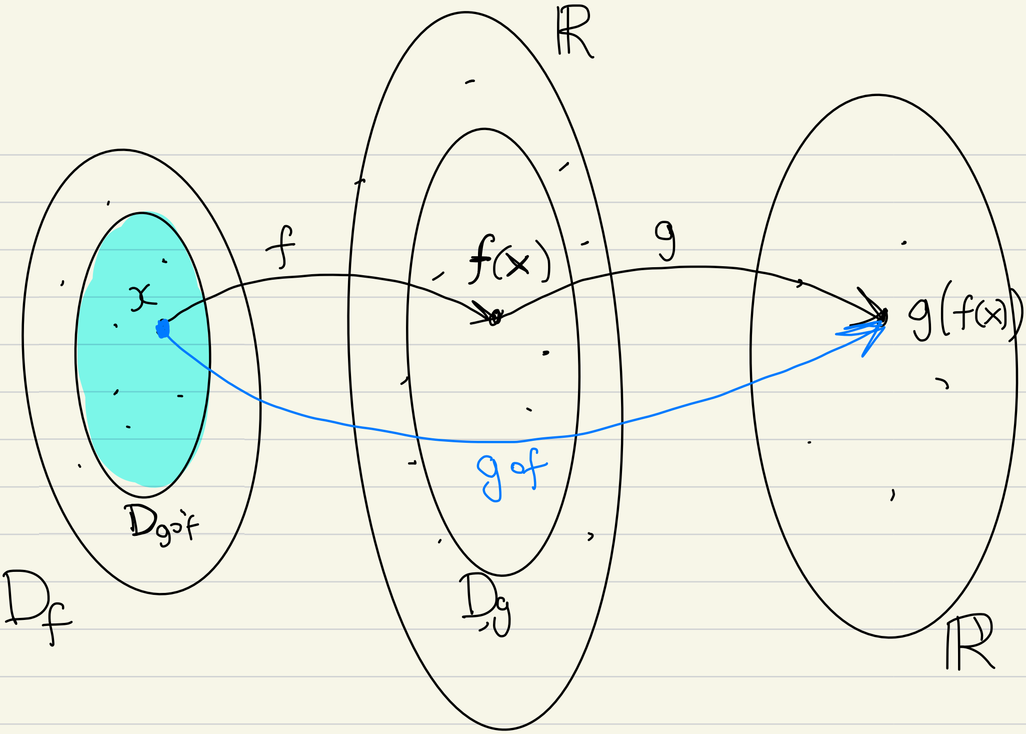


We define the composition of g with f to be the function $g \circ f$ with domain

$$D_{g \circ f} = \{ x \in D_f \;:\; f(x) \in D_g \}$$

and

$$(g \circ f)(x) = g(f(x)), \quad \text{for any } x \in D_{g \circ f}.$$

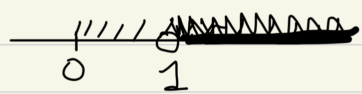


$$D_{g\circ f} = \{ x \in D_f : f(x) \in D_g \}.$$

Example : Let $f : [0,\infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$,
and $g : (1,+\infty) \to \mathbb{R}$, $g(x) = \ln(x-1)$
Find $g\circ f$.

- The domain of $g\circ f$ is

$$D_{g\circ f} = \{ x \in D_f : f(x) \in D_g \}$$
$$= \{ x \geqslant 0 : \sqrt{x} > 1 \}$$
$$= \{ x \geqslant 0 : x > 1 \}$$
$$= (1, +\infty).$$



$f(x) = \sqrt{x}$
$g(x) = \ln(x-1)$

For any $x \in D_{g \circ f} = (1, +\infty)$,

$$(g \circ f)(x) = g(f(x))$$
$$= \ln(f(x) - 1)$$
$$= \ln(\sqrt{x} - 1).$$

So

$g \circ f : (1, +\infty) \to \mathbb{R}$, $(g \circ f)(x) = \ln(\sqrt{x} - 1)$.

Exercise: Consider the functions

$$f : (-\infty, 0) \cup (0, +\infty) \to \mathbb{R}, \quad f(x) = \frac{1}{x}$$

$$g : \mathbb{R} \to \mathbb{R}, \quad g(x) = x^2 - 1.$$

(i) Find $g \circ f$.
(ii) Find $f \circ g$.
(iii) Are the functions $g \circ f$, $f \circ g$ equal?

(i) The domain of $g \circ f$ is

$$\begin{aligned} D_{g \circ f} &= \{ x \in D_f : f(x) \in D_g \} \\ &= \{ x \neq 0 : \tfrac{1}{x} \in \mathbb{R} \} \\ &= (-\infty, 0) \cup (0, +\infty). \end{aligned}$$

For any $x \in D_{g \circ f}$,

$$\begin{aligned} (g \circ f)(x) &= g(f(x)) \\ &= f^2(x) - 1 = \frac{1}{x^2} - 1. \end{aligned}$$

(ii) $D_{f\circ g} = \{ x \in D_g : g(x) \in D_f \}$

$= \{ x \in \mathbb{R} : x^2 - 1 \neq 0 \}$

$= \{ x \in \mathbb{R} : x \neq \pm 1 \}$

$= \mathbb{R} \setminus \{\pm 1\}$

$= (-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$.

For any $x \in D_{f\circ g}$,

$$(f\circ g)(x) = f(g(x)) = \frac{1}{g(x)} = \frac{1}{x^2-1}.$$

(iii) NO : $D_{g\circ f} \neq D_{f\circ g}$.

Exercise: With f, g as previously, find the functions

$f \circ f$ and $g \circ g$.

INVERSE FUNCTIONS

Let $f : A \to \mathbb{R}$. We say that f is <u>1 - 1</u> ("one-to-one") or <u>injective</u> if for <u>any</u> $x_1, x_2 \in A$,

$$x_1 \neq x_2 \implies f(x_1) \neq f(x_2).$$

(I.e. a function f is 1-1 if different elements $x_1, x_2 \in A$ are mapped onto diffcent numbers in $\mathbb{R}$).

An equivalent definition: $f : A \to \mathbb{R}$ is 1-1 if for any $x_1, x_2 \in A$, we have

$$f(x_1) = f(x_2) \implies x_1 = x_2.$$