We have seen that when f is continuous, then

$$\left(\int_a^x f(t)\,dt\right)' = f(x).$$

Recall: $\left(f(g(x))\right)' = f'(g(x)) \cdot g'(x)$

Exercise: Find the derivatives of:

(i) $F(x) = \int_0^{x^2} (u^3 - 2)\,du$

(ii) $G(x) = \int_{\sin x}^{1} t^2\,dt$

(iii) $H(x) = \int_{x^2}^{x^3} \frac{e^t}{t}\,dt$

ANSWER

(i) Let $g(x) = \int_0^x (u^3 - 2)\,du$.

We know that $g'(x) = x^3 - 2$.

The function in question is

$$F(x) = g(x^2) \Rightarrow F'(x) = g'(x^2) \cdot (x^2)'$$
$$= (x^6 - 2) \cdot 2x$$
$$= 2x(x^6 - 2) = 2(x^7 - 2x).$$

(ii) $$G(x) = \int_{\sin x}^{1} t^2\,dt = -\int_{1}^{\sin x} t^2\,dt$$

$$G'(x) = -\left(\int_{1}^{\sin x} t^2\,dt\right)'$$

$$= -(\sin x)^2 \cdot (\sin x)'$$

$$= -\sin^2 x \cdot \cos x.$$

(iii) $$H(x) = \int_{x^2}^{x^3} \frac{e^t}{t}\,dt$$

$$= \int_{x^2}^{1} \frac{e^t}{t}\,dt + \int_{1}^{x^3} \frac{e^t}{t}\,dt$$

$$= \int_{1}^{x^3} \frac{e^t}{t}\,dt - \int_{1}^{x^2} \frac{e^t}{t}\,dt$$

so

$$H'(x) = \frac{e^{x^3}}{x^3} \cdot (x^3)' - \frac{e^{x^2}}{x^2} \cdot (x^2)'$$

$$= \frac{3x^2 e^{x^3}}{x^3} - \frac{2x e^{x^2}}{x^2} = \frac{3e^{x^3} - 2e^{x^2}}{x}.$$

These examples are all particular cases of a general theorem.

<u>LEIBNIZ'S RULE</u>: Suppose $g(x)$ and $h(x)$ are differentiable and for any x the function f is continuous on the interval defined by $g(x)$ and $h(x)$. Then

$$\left(\int_{g(x)}^{h(x)} f(t)\,dt \right)' = f(h(x))\cdot h'(x) - f(g(x))g'(x).$$

* In the integral $\int_a^b f(x)\,dx$,

the symbol "dx" is called the <u>differential</u> of the variable x.

When we have a variable y which is a function of x, say

$$y = f(x)$$

then the differential of y is

$$\boxed{dy = f'(x)\,dx}$$

E.g. if $y = x^2$ then $dy = 2x\,dx$

INTEGRATION BY SUBSTITUTION

If $f, g : [a,b] \to \mathbb{R}$ are continuous and g is differentiable, we have

$$\boxed{\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.}$$

In practice, when we see the integral

$$\int_a^b f(g(x))\, g'(x)\, dx$$

we set $u = g(x) \Rightarrow du = g'(x)\, dx$

$$x_1 = a \Rightarrow u_1 = g(a)$$
$$x_2 = b \Rightarrow u_2 = g(b)$$

and

$$\int_a^b f(g(x))\, \underbrace{g'(x)\, dx}_{du} = \int_{g(a)}^{g(b)} f(u)\, du$$

$$\bullet \int_0^1 (2x+1)\, e^{x^2+x}\, dx = \int_0^1 (x^2+x)'\, e^{x^2+x}\, dx$$

Set $u = x^2+x \Rightarrow du = (2x+1)dx$

$x_1 = 0 \Rightarrow u_1 = 0$

$x_1 = 1 \Rightarrow u_2 = 2$

$$= \int_0^2 e^u \, du$$

$$= \left[e^u\right]_0^2$$

$$= e^2 - 1 .$$

- $\displaystyle\int_{e}^{e^2} \frac{1}{x\ln x}\,dx =$

Set $u = \ln x \Rightarrow$
$du = \frac{1}{x}\,dx$

$x_1 = e \Rightarrow u_1 = 1$
$x_2 = e^2 \Rightarrow u_2 = 2$

$$= \int_1^2 \frac{1}{u}\,du = \left[\ln|u|\right]_1^2 = \ln 2.$$

- $\displaystyle\int_0^1 e^{3x+1}\,dx =$

Set $u = 3x+1$
$du = (3x+1)'dx = 3dx$

$x_1 = 0 \Rightarrow u_1 = 1$
$x_2 = 1 \Rightarrow u_2 = 4$

$$= \int_1^4 e^u \cdot \frac{1}{3} \cdot du = \frac{1}{3}\int_1^4 e^u\,du$$

$$= \frac{1}{3}\left[e^u\right]_1^4 = \frac{1}{3}(e^4 - e).$$

$du = 3\,dx \Rightarrow$

$dx = \frac{1}{3}\,du$

- $$\int_0^{\pi/2} \sin\left(\frac{x}{2}\right) dx =$$

Set $u = \frac{x}{2} \Rightarrow du = \frac{1}{2} dx$

$\Rightarrow dx = 2du$

$x_1 = 0 \Rightarrow u_1 = 0$
$x_2 = \frac{\pi}{2} \Rightarrow u_2 = \frac{\pi}{4}$

$$= \int_0^{\pi/4} \sin u \cdot 2\,du = 2\int_0^{\frac{\pi}{4}} \sin u\, du$$

$$= 2\left[-\cos u\right]_0^{\pi/4} = 2\left(-\cos\frac{\pi}{4} - (-\cos 0)\right)$$

$$= 2\left(-\frac{\sqrt{2}}{2} + 1\right) = 2\left(1 - \frac{\sqrt{2}}{2}\right) = 2 - \sqrt{2}.$$

$$u = \sqrt{x^2+1}$$
$$du = \frac{2x}{2\sqrt{x^2+1}}\,dx = \frac{x\,dx}{\sqrt{x^2+1}}$$

- $$\int_0^{2\sqrt{2}} 4x\sqrt{x^2+1}\,dx$$

Set $u = x^2+1$

$du = 2x\,dx$

$x_1 = 0 \Rightarrow u_1 = 1$

$x_2 = 2\sqrt{2} \Rightarrow u_2 = 9$

$$\int_0^{2\sqrt{2}} 4x\sqrt{x^2+1}\,dx = \int_0^{2\sqrt{2}} 2\cdot \underline{2x}\cdot\sqrt{x^2+1}\,\underline{dx}$$

$$= \int_1^9 2\sqrt{u}\,du$$

$$= \int_1^9 2u^{\frac{1}{2}}\,du$$

$$= \left[2\,\frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^9 = \left[2\,\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^9$$

$$= \frac{4}{3}\left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right)$$

$$= \frac{4}{3}\left(\sqrt{9}^{\,3} - 1\right) = \frac{4}{3}(27-1) = \frac{104}{3}.$$

$(x^3+x)' = 3x^2+1$

$$\int_1^2 \frac{3x^2+1}{x^3+x}\,dx =$$

Set $u = x^3+x$

$du = (3x^2+1)\,dx$

$x_1 = 1 \Rightarrow u_1 = 2$

$x_2 = 2 \Rightarrow u_2 = 10$

$$= \int_2^{10} \frac{1}{u}\,du = \left[\ln|u|\right]_2^{10}$$

$$= \ln 10 - \ln 2$$

$$= \ln 5.$$

- $\displaystyle\int_{1/2}^{1} \frac{e^{1/x}}{x^2}\,dx =$

Set $u = \frac{1}{x}$

$du = \left(\frac{1}{x}\right)' dx = -\frac{1}{x^2}\,dx$

$x_1 = \frac{1}{2} \Rightarrow u_1 = 2$

$x_2 = 1 \Rightarrow u_2 = 1$

$$= \int_{2}^{1} -e^{u}\,du = \int_{1}^{2} e^{u}\,du = \left[e^{u}\right]_1^2 = e^2 - e.$$

- $\displaystyle\int_{0}^{\pi/6} \cos x\, e^{\sin x}\,dx =$

Set $u = \sin x$

$du = \cos x\,dx$

$x = 0 \Rightarrow u = 0$

$x = \frac{\pi}{6} \Rightarrow u = \sin\frac{\pi}{6} = \frac{1}{2}$

$$= \int_{0}^{1/2} e^{u}\,du = \left[e^{u}\right]_0^{1/2} = \sqrt{e} - 1.$$

$$\int_1^2 \frac{1}{y}\,dy = \left[\ln|y|\right]_1^2 = \ln 2$$

- $$\int_4^9 \frac{2}{x-3}\,dx =$$

Set $u = x-3 \Rightarrow du = dx$

$x = 4 \Rightarrow u = 1$

$x = 9 \Rightarrow u = 6$

$$= \int_1^6 \frac{2}{u}\,du = 2\left[\ln|u|\right]_1^6 = 2\ln 6.$$

- $$\int_1^4 \frac{dx}{2x+1} =$$

Set $u = 2x+1 \Rightarrow du = 2\,dx$

$x = 1 \Rightarrow u = 3$

$x = 4 \Rightarrow u = 9$

$$= \int_3^9 \frac{1}{u}\cdot\frac{1}{2}\,du = \frac{1}{2}\int_3^9 \frac{1}{u}\,du$$

$$= \frac{1}{2}\left[\ln|u|\right]_3^9 = \frac{\ln 3}{2}.$$

- $$\int_{-2}^{-1} \frac{dx}{2x+5} =$$

Set $u = 2x+5$
$du = 2dx$

$x = -2 \Rightarrow u = 1$
$x = -1 \Rightarrow u = 3$

$$= \int_{1}^{3} \frac{1}{2u}\, du = \frac{1}{2} \ln 3.$$

- $$\int_{-3}^{-1} \frac{dx}{x} \overset{!!!}{=} \Big[\ln|x|\Big]_{-3}^{-1} = \ln 1 - \ln 3$$

$$= \ln \frac{1}{3}.$$

(we don't need a substitution).