NTNU
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Department of Mathematical Sciences

Examination paper for **MA0001 Brukerkurs i Matematikk A**

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Problem 1 Find the derivative of the function

$$f(x) = \frac{e^{2x}}{x^2+1}, \qquad x \in \mathbb{R}.$$

Solution We have

$$\begin{aligned} f'(x) &= \left(\frac{e^{2x}}{x^2+1}\right)' \\ &= \frac{(e^{2x})'(x^2+1) - e^{2x}(x^2+1)'}{(x^2+1)^2} \\ &= \frac{2e^{2x}(x^2+1) - 2xe^{2x}}{(x^2+1)^2} \\ &= \frac{2(x^2-x+1)e^{2x}}{(x^2+1)^2}. \end{aligned}$$

Problem 2 Find the equation of the tangent line of the graph of the function

$$g(x) = x\sin x + x^2, \qquad x \in \mathbb{R}$$

at the point $(0, g(0))$.

Solution We have $g(0) = 0$. Also

$$g'(x) = (x\sin x)' + (x^2)' = \sin x + x\cos x + 2x$$

and therefore $g'(0) = 0$. The equation of the tangent line is thus

$$y = g'(0)(x-0) + g(0) \quad \Rightarrow \quad y = 0.$$

Problem 3 Let

$$f(x) = xe^{-x} + 1, \qquad x \in \mathbb{R}.$$

Find the second degree Taylor polynomial of f at the point $x_0 = 0$.

Solution We begin by finding the first and second derivative of f. We have

$$\begin{aligned} f(x) &= xe^{-x} + 1, \\ f'(x) &= (xe^{-x})' + (1)' = e^{-x} - xe^{-x} = (1-x)e^{-x}, \\ f''(x) &= (1-x)'e^{-x} + (1-x)(e^{-x})' \\ &= -e^{-x} - (1-x)e^{-x} = (x-2)e^{-x}. \end{aligned}$$

The 2nd degree Taylor polynomial of f at 0 is

$$P(x) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2.$$

Since

$$f(0) = 1, f'(0) = 1, f''(0) = -2,$$

this polynomial is equal to

$$P(x) = 1 + x - x^2.$$

Problem 4 Evaluate the integral

$$\int_0^1 \left(2^x + \sqrt{x}\right) \mathrm{d}x.$$

Solution The integral is

$$\begin{aligned} \int_0^1 (2^x + \sqrt{x})\mathrm{d}x &= \int_0^1 (2^x + x^{1/2})\mathrm{d}x = \left[\frac{2^x}{\ln 2} + \frac{x^{3/2}}{3/2}\right]_0^1 \\ &= \left[\frac{2^x}{\ln 2} + \frac{2}{3}x^{3/2}\right]_0^1 = \left(\frac{2}{\ln 2} + \frac{2}{3}\right) - \left(\frac{1}{\ln 2} + 0\right) \\ &= \frac{1}{\ln 2} + \frac{2}{3}. \end{aligned}$$

Problem 5 Find the equation of the tangent line of the hyperbola

$$\frac{y^2}{2} - x^2 = 1$$

at the point $(1, 2)$.

Solution Implicit differentiation gives

$$yy' - 2x = 0 \quad \Rightarrow \quad y' = \frac{2x}{y}\cdot$$

At the point $(1, 2)$, i.e. when $x = 1$ and $y = 2$, this gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1.$$

The tangent line of the hyperbola at $(1, 2)$ has slope equal to 1, and therefore its equation is

$$y - 2 = 1 \cdot (x - 1) \quad \Rightarrow \quad y = x + 1.$$

Problem 6 Find the limit

$$\lim_{x \to 0} \frac{2\cos x + x^2 - 2}{x^4}\cdot$$

Solution We need to apply the rule of de l'Hospital successively.

$$\begin{aligned}
\lim_{x \to 0} \frac{2\cos x + x^2 - 2}{x^4} &\overset{(\frac{0}{0})}{=} \lim_{x \to 0} \frac{(2\cos x + x^2 - 2)'}{(x^4)'} = \lim_{x \to 0} \frac{2x - 2\sin x}{4x^3} \\
&\overset{(\frac{0}{0})}{=} \lim_{x \to 0} \frac{(2x - 2\sin x)'}{(4x^3)'} = \lim_{x \to 0} \frac{2 - 2\cos x}{12x^2} = \lim_{x \to 0} \frac{1 - \cos x}{6x^2} \\
&\overset{(\frac{0}{0})}{=} \lim_{x \to 0} \frac{(1 - \cos x)'}{(6x^2)'} = \lim_{x \to 0} \frac{\sin x}{12x} \\
&\overset{(\frac{0}{0})}{=} \lim_{x \to 0} \frac{(\sin x)'}{(12x)'} = \lim_{x \to 0} \frac{\cos x}{12} = \frac{1}{12}.
\end{aligned}$$

Problem 7 Prove that the equation

$$x^7 + 3x^4 - x^2 - 2 = 0$$

has at least one solution in the interval $(0, 1)$. Justify your answer.

Solution Set

$$f(x) = x^7 + 3x^4 - x^2 - 2.$$

The given equation is then equivalent to

$$f(x) = 0.$$

We need to observe the following:
- f is continuous on $[0, 1]$ (because it is a polynomial),
- $f(0) = -2 < 0$ and $f(1) = 1 + 3 - 1 - 2 = 1 > 0$.

Therefore by the Intermediate value Theorem, there exists $x_0 \in (0, 1)$ such that $f(x_0) = 0$.

Problem 8 Let

$$h(x) = 2x^3 + 3x^2 - 12x + 5, \qquad x \in \mathbb{R}.$$

Find the intervals where h is monotonic. Find all global and local extrema of h (maxima and minima).

Solution The first derivative of h is

$$h'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x + 2)(x - 1).$$

We have

$$h'(x) > 0 \Leftrightarrow (x + 2)(x - 1) > 0 \Leftrightarrow x \in (-\infty, -2) \cup (1, +\infty).$$
$$h'(x) < 0 \Leftrightarrow (x + 2)(x - 1) < 0 \Leftrightarrow x \in (-2, 1).$$

Therefore the function h is increasing on the intervals $(-\infty, -2]$ and $[1, +\infty)$ and decreasing on $[-2, 1]$.

We have

$$\lim_{x \to -\infty} h(x) = -\infty, \qquad h(-2) = 25, \qquad h(1) = -2, \qquad \lim_{x \to +\infty} h(x) = +\infty.$$

x	$-\infty$		-2		1		$+\infty$
$h'(x)$		$+$	0	$-$	0	$+$	
$h(x)$	$-\infty$	$\nearrow$	$h(-2) = 25$	$\searrow$	$h(1) = -2$	$\nearrow$	$+\infty$

The function h has a local maximum at -2 which is equal to $h(-2) = 25$, a local minimum at 1 which is equal to $h(1) = -2$, while it does not have any global maxima or minima.

Problem 9 Find the value of $a > 0$ such that

$$\int_0^\infty e^{-ax}\mathrm{d}x = 1.$$

Solution We first find the value of the improper integral. For $T > 0$,

$$\begin{aligned}\int_0^T e^{-ax}\mathrm{d}x &= \left[\frac{e^{-ax}}{-a}\right]_0^T \\ &= \frac{1}{a}(1 - e^{-aT}) \\ &= \frac{1}{a}\left(1 - \frac{1}{e^{aT}}\right).\end{aligned}$$

Since

$$\lim_{T\to\infty}\int_0^T e^{-ax}\mathrm{d}x = \lim_{T\to\infty}\frac{1}{a}\left(1 - \frac{1}{e^{aT}}\right) = \frac{1}{a},$$

the improper integral

$$\int_0^\infty e^{-ax}\mathrm{d}x = \frac{1}{a}\cdot$$

Therefore we have

$$\int_0^\infty e^{-ax}\mathrm{d}x = 1 \quad \Leftrightarrow \quad \frac{1}{a} = 1 \quad \Leftrightarrow \quad a = 1.$$

Problem 10 Find the area of the bounded domain of the 2–dimensional plane that is surrounded by the graph of the function $f(x) = x^2$, the line with equation $y = 2 - x$ and the horizontal axis. Draw a sketch.

Solution We first find the point of intersection of the two curves.

$$\begin{aligned} x^2 = 2 - x &\Leftrightarrow x^2 + x - 2 = 0 \\ &\Leftrightarrow (x+2)(x-1) = 0 \\ &\Leftrightarrow x = -2 \text{ or } x = 1. \end{aligned}$$

Here the point of intersection which is of interest is $(1, 1)$. We find the area in question as the sum of the areas of two smaller domains. The first is the one between the graph of $f(x) = x^2$, the vertical line $x = 1$ and the horizontal axis, which has area

$$A_1 = \int_0^1 x^2 \mathrm{d}x = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

The second is the domain surrounded by the line $x = 1$, the line $y = 2 - x$ and the horizontal axis; this has area

$$A_2 = \int_1^2 (2 - x)\mathrm{d}x = \left[2x - \frac{x^2}{2}\right]_1^2 = \frac{1}{2}.$$

(Alternatively, this area can be calculated as the area of a right triangle with perpendicular sides of lengths both equal to 1.) Therefore the area in question is

$$A = A_1 + A_2 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$