NTNU
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Department of Mathematical Sciences

Examination paper for **MA0001 Brukerkurs i Matematikk A**

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Problem 1 Find the equation of the tangent line of the graph of the function

$$f(x) = x^2 e^x + x, \qquad x \in \mathbb{R}$$

at the point $(0, f(0))$.

Solution. We have

$$f'(x) = 2xe^x + x^2 e^x + 1 = (x^2 + 2x)e^x + 1.$$

Thus $f(0) = 0$, $f'(0) = 1$ and the equation of the tangent line of the graph of f at the point $(0, f(0))$ is

$$y - f(0) = f'(0)(x - 0) \qquad \Rightarrow \qquad y = x.$$

Problem 2 Find the derivative of the function

$$g(x) = (1 + \ln x)^3, \qquad x \in (0, +\infty).$$

Solution. We have

$$\begin{aligned} g'(x) &= 3(1 + \ln x)^2 \cdot (1 + \ln x)' \\ &= \frac{3(1 + \ln x)^2}{x}. \end{aligned}$$

Problem 3 Let

$$h(x) = x \cos x + x, \qquad x \in \mathbb{R}.$$

Find the second degree Taylor polynomial of h at the point $x_0 = 0$.

Solution We have

$$\begin{aligned} h'(x) &= \cos x - x \sin x + 1, \\ h''(x) &= -\sin x - \sin x - x \cos x = -2 \sin x - x \cos x \end{aligned}$$

therefore $h(0) = 0, h'(0) = 2, h''(0) = 0$ and the polynomial is

$$P(x) = h(0) + \frac{h'(0)}{1!}(x - 0) + \frac{h''(0)}{2!}(x - 0)^2 = 2x.$$

Problem 4 Evaluate the integral

$$\int_0^{\ln 2} xe^x \mathrm{d}x.$$

Solution We have

$$\begin{aligned}\int_0^{\ln 2} xe^x \mathrm{d}x &= \int_0^{\ln 2} x(e^x)' \mathrm{d}x = [xe^x]_0^{\ln 2} - \int_0^{\ln 2} (x)' e^x \mathrm{d}x \\ &= \ln 2 \cdot e^{\ln 2} - \int_0^{\ln 2} e^x \mathrm{d}x \\ &= 2\ln 2 - [e^x]_0^{\ln 2} \\ &= 2\ln 2 - (e^{\ln 2} - 1) \\ &= 2\ln 2 - 1.\end{aligned}$$

Problem 5 Use implicit differentiation to find the equation of the tangent line of the ellipse

$$\frac{x^2}{2} + \frac{y^2}{4} = 1$$

at its point $(-1, \sqrt{2})$.

Solution Implicit differentiation gives

$$\frac{x^2}{2} + \frac{y^2}{4} = 1 \quad \Rightarrow \quad x + \frac{yy'}{2} = 0 \quad \Rightarrow \quad y' = -\frac{2x}{y}.$$

For $x = -1, y = \sqrt{2}$, this gives $y' = \sqrt{2}$. Therefore the tangent line of the ellipse at the point $(-1, \sqrt{2})$ is

$$y - \sqrt{2} = \sqrt{2}(x + 1) \quad \Rightarrow \quad y = \sqrt{2}x + 2\sqrt{2}.$$

Problem 6 Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = -1$ and $f(1) = 2$. Prove that there exists some $x_0 \in (0,1)$ such that

$$f(x_0) = x_0^2.$$

Solution Consider the function

$$g : [0,1] \to \mathbb{R}, \qquad g(x) = f(x) - x^2.$$

We see that g is continuous (since it is the difference of two continuous functions) and also

$$g(0) = f(0) = -1 < 0, \qquad g(1) = f(1) - 1 = 1 > 0.$$

Therefore by the intermediate value theorem there exists some $x_0 \in (0,1)$ such that

$$g(x_0) = 0 \qquad \Leftrightarrow \qquad f(x_0) = x_0^2.$$

Problem 7 Find the limit

$$\lim_{x \to 0} \frac{e^{x^2} - 1}{x^2}.$$

Solution We use L'Hospital's rule.

$$\lim_{x \to 0} \frac{e^{x^2} - 1}{x^2} \overset{(\frac{0}{0})}{=} \lim_{x \to 0} \frac{2xe^{x^2}}{2x} = \lim_{x \to 0} e^{x^2} = 1.$$

Problem 8 Let

$$f(x) = \frac{\ln x}{x}, \qquad x > 0.$$

Find the intervals where f is monotonic and where it is convex and concave. Find all global and local extrema of f (maxima and minima). What are the asymptotes of the graph of f?

Solution We have

$$f'(x) = \frac{(\ln x)'x - \ln x(x)'}{x^2} = \frac{1 - \ln x}{x^2}$$

and

$$f''(x) = \frac{(1 - \ln x)'x^2 - 2x(1 - \ln x)}{x^4} = -\frac{3 - 2\ln x}{x^3}.$$

Thus

$$f'(x) > 0 \Leftrightarrow x < e \qquad \text{and} \qquad f'(x) < 0 \Leftrightarrow x > e$$

while

$$f''(x) > 0 \Leftrightarrow x > e^{3/2} \qquad \text{and} \qquad f''(x) < 0 \Leftrightarrow x < e^{3/2}.$$

This means that f is strictly increasing on $(0, e]$ and strictly decreasing on $[e, +\infty)$; it is concave on $(0, e^{3/2}]$ and convex on $[e^{3/2}, +\infty)$. Also f has a global maximum at $x = e$, which is equal to $f(e) = 1/e$. Since $\lim_{x \to 0^+} f(x) = -\infty$, f has no global minimum.

Since the domain of definition of f is $(0, +\infty)$, the only possible asymptotes of the graph of f can be at 0 and $+\infty$. We have

$$\lim_{x \to 0^+} \frac{\ln x}{x} = -\infty \qquad \text{and} \qquad \lim_{x \to +\infty} \frac{\ln x}{x} = \lim_{x \to +\infty} \frac{1}{x} = 0,$$

therefore the asymptotes of the graph of f are $x = 0$ and $y = 0$.

Problem 9 Evaluate the integral

$$\int_1^\infty \frac{1}{x^3} \mathrm{d}x.$$

Solution For any $T > 0$ we have

$$\int_1^T \frac{1}{x^3}\mathrm{d}x = \int_1^T x^{-3}\mathrm{d}x = \left[\frac{x^{-2}}{-2}\right]_1^T = \frac{1}{2} - \frac{1}{2T^2}$$

therefore

$$\lim_{T\to\infty}\int_1^T \frac{1}{x^3}\mathrm{d}x = \lim_{T\to\infty}\left(\frac{1}{2} - \frac{1}{2T^2}\right) = \frac{1}{2}$$

and

$$\int_1^\infty \frac{1}{x^3}\mathrm{d}x = \frac{1}{2}.$$

Problem 10 Let $h : [0,3] \to \mathbb{R}$ be a continuous function. Also let

$$f : [0,3] \to \mathbb{R}, \qquad f(x) = h(x) + x^2 + 2,$$
$$g : [0,3] \to \mathbb{R}, \qquad g(x) = h(x) + 3x.$$

Find the area between the graphs of the functions f and g.

Solution We first need to know whether $f(x)$ is bigger than $g(x)$ or the opposite is true. We have

$$\begin{aligned} f(x) > g(x) &\Leftrightarrow h(x) + x^2 + 2 > h(x) + 3x \\ &\Leftrightarrow x^2 - 3x + 2 > 0 \\ &\Leftrightarrow (x-1)(x-2) > 0 \\ &\Leftrightarrow x < 1 \text{ or } x > 2 \end{aligned}$$

and similarly

$$f(x) < g(x) \qquad \Leftrightarrow \qquad 1 < x < 2.$$

Therefore the requested area is equal to

$$\begin{aligned} &\int_0^1 (f(x) - g(x))\mathrm{d}x + \int_1^2 (g(x) - f(x))\mathrm{d}x + \int_2^3 (f(x) - g(x))\mathrm{d}x = \\ &= \int_0^1 (x^2 - 3x + 2)\mathrm{d}x + \int_1^2 (3x - x^2 - 2)\mathrm{d}x + \int_2^3 (x^2 - 3x + 2)\mathrm{d}x \\ &= \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_0^1 + \left[\frac{3x^2}{2} - \frac{x^3}{3} - 2x\right]_1^2 + \left[\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right]_2^3 \\ &= \frac{5}{6} + \frac{1}{6} + \frac{5}{6} = \frac{11}{6}. \end{aligned}$$

Remark: *Some students calculated the area between the graphs of f and g and the lines $x = 1, x = 2$ which is equal to $1/6$. This answer will also be considered as correct.*