Forklaring på siste eksempel brukt i forelesnignen 14. November, 2017.

1. Regn ut:
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx.$$

Løsning: Vi splitter det gitte integralet i to ledd:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{-\infty}^{0} \frac{1}{1+x^2} dx + \int_{0}^{\infty} \frac{1}{1+x^2} dx.$$

Løser første ledd:

$$\begin{aligned}
\int_{-\infty}^{0} \frac{1}{1+x^2} dx &= \lim_{z\to-\infty} \int_{z}^{0} \frac{1}{1+x^2} dx \\
&= \lim_{z\to-\infty} [\tan^{-1}(x)]_z^0 \qquad \left[\text{fordi } \int \frac{1}{1-x^2} = \tan^{-1}(x) + C \right] \\
&= \lim_{z\to-\infty} [\tan^{-1}(0) - \tan^{-1}(z)] \\
&= [\tan^{-1}(0) - \tan^{-1}(-\infty)] \\
&= [\tan^{-1}(\tan 0) - \tan^{-1}(-\tan\frac{\pi}{2})] \qquad \left[\text{fordi } \tan^{-1}(\frac{\pi}{2}) = \frac{\sin^{-1}(\frac{\pi}{2})}{\cos^{-1}(\frac{\pi}{2})} = \frac{1}{0} = \infty \right] \\
&= [0 + \tan^{-1}(\tan\frac{\pi}{2})] \qquad [\text{ fordi } \tan^{-1}(-x) = -\tan^{-1}(x)] \\
&= [0 + (\frac{\pi}{2})] = \frac{\pi}{2}
\end{aligned}$$

Dermed får vi:

$$\int_{-\infty}^{0} \frac{1}{1+x^2} dx = \frac{\pi}{2}.$$

For andre ledd:

$$\begin{aligned}
\int_{0}^{\infty} \frac{1}{1+x^2} dx &= \lim_{z\to\infty} \int_{0}^{z} \frac{1}{1+x^2} dx \\
&= \lim_{z\to\infty} [\tan^{-1}(x)]_0^z \\
&= \lim_{z\to\infty} [\tan^{-1}(z) - \tan^{-1}(0)] \\
&= [\tan^{-1}(\infty) - \tan^{-1}(0)] \\
&= [\tan^{-1}(\tan\frac{\pi}{2}) - \tan^{-1}(\tan 0)] \\
&= [(\frac{\pi}{2}) - 0] = \frac{\pi}{2}
\end{aligned}$$

Dermed får vi:

$$\int_0^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2}.$$

Og så:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$