IMAT2100 Matematiske metoder 3

Fredag 15/10-21 08:15-10:00

Eksempel: (Bytte av integrasjonsrekkefølge i trippelintegral).

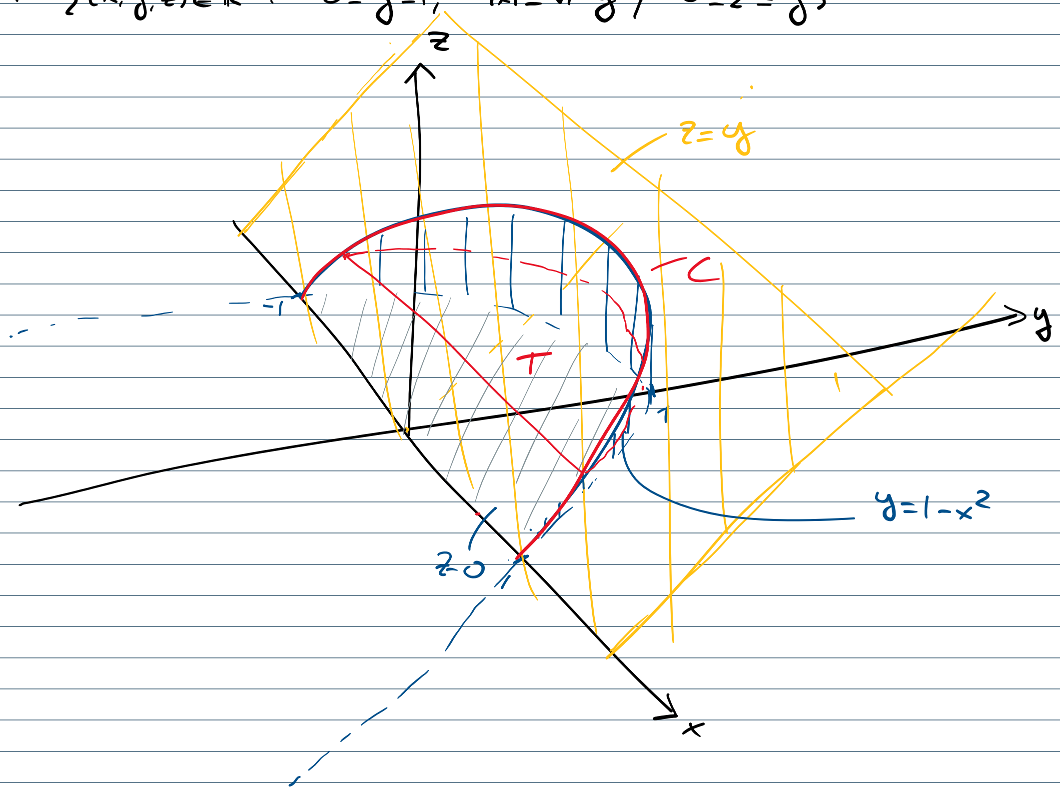
Et integral over et legeme T kan uttrykkes som det itererte integralet

$$I = \int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \int_0^y f(x,y,z)\,dz\,dx\,dy \qquad z \to x \to y$$

a) Skisser og beskriv legemet.

b) Bytt om rekkefølgen $y \to x \to z$ $\quad x^2 \le 1-y$ (NB: $y \le 1$)

a) $T = \{(x,y,z) \in \mathbb{R}^3 : \overbrace{0 \le y \le 1,\ |x| \le \sqrt{1-y}}^{\text{Projeksjon i yx-planet}},\ 0 \le z \le y\}$



Vi ser at T er kilen begrenset av xy-planet, planet $z=y$ og den parabolske sylinderen $y=1-x^2$

b) På kurven C $\quad z = y \le 1 \quad$ så $\quad 0 \le z \le 1$

For en fiksert z $\quad |x| \le \sqrt{1-y} \le \sqrt{1-z}$

For fiksert z og x $\quad z \le y \le 1-x^2$

$$T = \{(x,y,z) \in \mathbb{R}^3 : \underbrace{0 \le z \le 1,\ |x| \le \sqrt{1-z}},\ z \le y \le 1-x^2\}$$

Projeksjonen i xz-planet

og

$$I = \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_z^{1-x^2} f(x,y,z)\,dy\,dz\,dx \qquad z \to y \to x$$

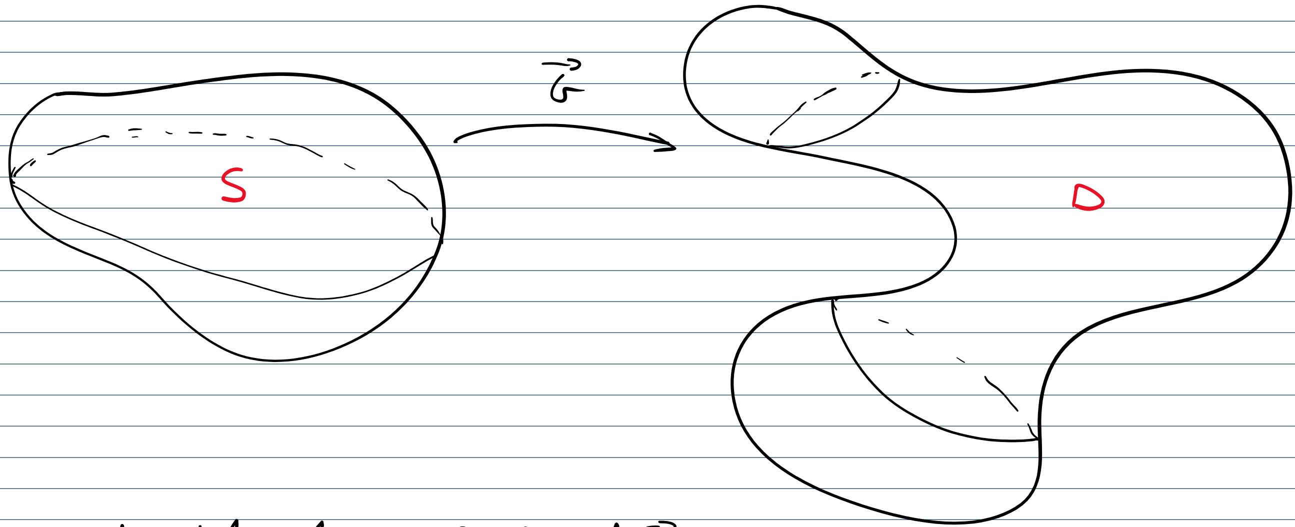
c) $z \to y \to x$

$$I = \int_{-1}^{1} \int_0^{1-x^2} \int_0^{y} f(x,y,z)\,dz\,dy\,dx \qquad \text{(Prøv selv)}$$

Variabelbytte i trippelintegraler (15.6)

$(x,y,z) = \vec{G}(u,v,w)$

Vi ønsker å integrere $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ ved å utføre substitusjon $\vec{x} = \vec{G}(\vec{u})$ med en kontinuerlig deriverbar én-til-én transformasjon $\vec{G} : S \to \mathbb{R}^3$ med $\vec{G}(S) = D$



Teorem: Om f er integrerbar på D så er $f \circ \vec{G}$ integrerbar på S og

$$\iiint_D f(\vec{x})\,dV_{\vec{x}} = \iiint_S f(\vec{G}(\vec{u}))\,\left|\det(D\vec{G}(\vec{u}))\right|\,dV_{\vec{u}} \qquad (*)$$

Hvis $\vec{x} = (x,y,z)$, $\vec{u} = (u,v,w)$ og $\vec{G}(\cdot) = (x(\cdot), y(\cdot), z(\cdot))$ blir $*$

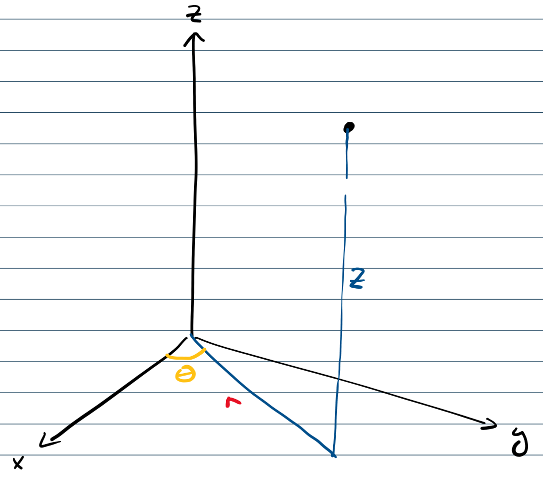
$(\cdot(u,v,w))$

$$\iiint f(x,y,z)\,dx\,dy\,dz = \iiint f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

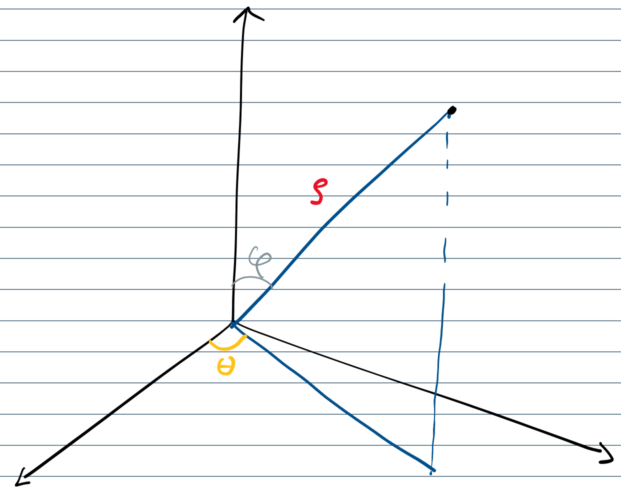
D S

Vi har lært om to ulike koordinatsystemer som alternativ til de kartesiske.

Sylinderkoordinater: $\vec{G}(r,\theta,z) = (r\cos(\theta),\ r\sin(\theta),\ z)$



Kulekoordinater $\vec{H}(\rho,\varphi,\theta) = (\rho\sin(\varphi)\cos(\theta),\ \rho\sin(\varphi)\sin(\theta),\ \rho\cos(\varphi))$



Teorem:

Vi har $dx\,dy\,dz = \begin{cases} r\,dr\,d\theta\,dz & \text{: sylinderkoordinater} \\ \rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta & \text{: kulekoordinater} \end{cases}$

Bevis:

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos(\theta) & \sin(\theta) & 0 \\ -r\sin(\theta) & r\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\cdot\begin{vmatrix} \cos(\theta) & \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) \end{vmatrix} = r$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \begin{vmatrix} \sin(\varphi)\cos(\theta) & \sin(\varphi)\sin(\theta) & \cos(\varphi) \\ \rho\cos(\varphi)\cos(\theta) & \rho\cos(\varphi)\sin(\theta) & -\rho\sin(\varphi) \\ -\rho\sin(\varphi)\sin(\theta) & \rho\sin(\varphi)\cos(\theta) & 0 \end{vmatrix}$$

$$= \rho^2\sin(\varphi)\begin{vmatrix} \sin(\varphi)\cos(\theta) & \sin(\varphi)\sin(\theta) & \cos(\varphi) \\ \cos(\varphi)\cos(\theta) & \cos(\varphi)\sin(\theta) & -\sin(\varphi) \\ -\sin(\theta) & \cos(\theta) & 0 \end{vmatrix}$$

$$= \rho^2\sin(\varphi)\underbrace{\left(\cos(\varphi)^2 \underbrace{\begin{vmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{vmatrix}}_{\cos(\theta)^2+\sin(\theta)^2=1} + \sin(\varphi)^2 \underbrace{\begin{vmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{vmatrix}}_{=1}\right)}_{\cos(\varphi)^2+\sin(\varphi)^2=1}$$

$$= \rho^2\sin(\varphi)$$

<u>Eksempler:</u>

① Volumet av en kule med radius R.

$$B_R = \{\vec{x}\in\mathbb{R}^3 : |\vec{x}| \le R\} = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2 \le R^2\}$$

Svarer til boksen $\overset{\rho}{[0,R]} \times \overset{\varphi}{[0,\pi]} \times \overset{\theta}{[0,2\pi]}$

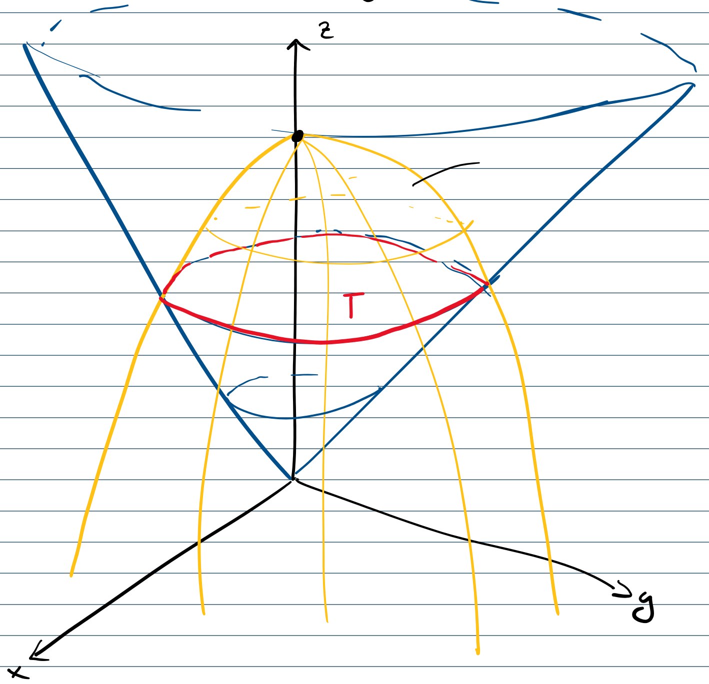
$$|B_R| = \iiint_{B_R} dx\,dy\,dz = \int_0^R\int_0^\pi\int_0^{2\pi} 1\cdot\rho^2\sin(\varphi)\,d\theta\,d\varphi\,d\rho$$

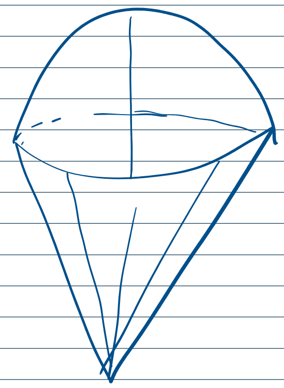
Separabel integrand

$$= \left(\int_0^{2\pi} 1\,d\theta\right)\left(\int_0^R \rho^2\,d\rho\right)\left(\int_0^\pi \sin(\varphi)\,d\varphi\right) \qquad \Big[-\cos(\varphi)\Big]_0^\pi$$

$$= 2\pi\cdot\frac{1}{3}R^3\cdot 2 = \frac{4\pi}{3}R^3$$

② Ta legemet som ligger inni kjeglen $\frac{z^2}{3} = x^2+y^2$, over xy-planet, og under paraboloiden $2z = 1-(x^2+y^2)$





Hva blir $$\iiint_T \frac{1}{x^2+y^2+z^2}\,dx\,dy\,dz$$

Her er det veldig naturlig å bruke kulekoordinater

Kjeglen: $\frac{z^2}{3}=x^2+y^2$ $\qquad \frac{\cancel{\rho^2}\cos(\varphi)^2}{3} = r^2 = \cancel{\rho^2}\sin(\varphi)^2$

$$\tan(\varphi)^2=\frac{1}{3} \quad \text{eller} \quad \tan(\varphi)=\frac{1}{\sqrt{3}} \quad \text{eller} \quad \varphi=\frac{\pi}{6}$$

Paraboloiden: $2z=1-(x^2+y^2)$ $\qquad 2\rho\cos(\varphi)=1-\rho^2\sin(\varphi)^2$

$$\rho^2\sin(\varphi)^2+2\rho\cos(\varphi)-1=0$$

$$\rho = \frac{-2\cos(\varphi)+\sqrt{4\cos(\varphi)^2+4\sin(\varphi)^2}}{2\sin(\varphi)^2} = \frac{-\cos(\varphi)+1}{\sin(\varphi)^2}$$

$$= \frac{1}{\sin(\varphi)}\cdot\frac{\cancel{2}\sin\left(\frac{\varphi}{2}\right)^{\cancel{2}}}{\cancel{2\sin(\varphi/2)}\cos(\varphi/2)} = \frac{\tan(\varphi/2)}{\sin(\varphi)}$$

$$I = \int_0^{2\pi}\int_0^{\pi/6}\int_0^{\frac{\tan(\varphi/2)}{\sin(\varphi)}} \frac{1}{\cancel{\rho^2}}\cancel{\rho^2}\sin(\varphi)\,d\rho\,d\varphi\,d\theta = 2\pi\int_0^{\pi/6}\frac{\sin(\varphi/2)}{\cos(\varphi/2)}\,d\varphi$$

$\frac{\sin(\varphi/2)}{\cos(\varphi/2)}$

$u=\cos(\varphi/2)$

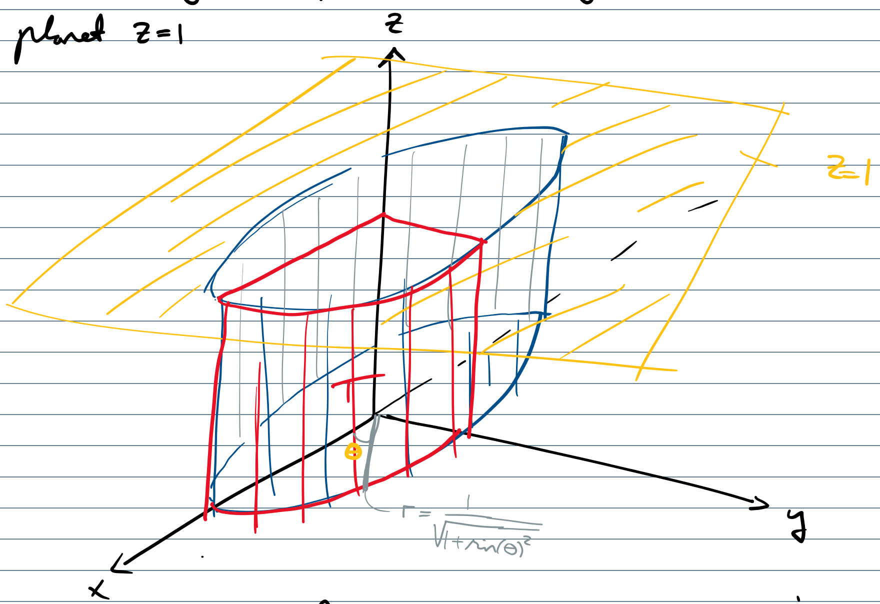
$du=-\frac{1}{2}\sin(\varphi/2)\,d\varphi$

$\sin(\varphi/2)\,d\varphi=2du$

1

$$= 4\pi \int_{\frac{1+\sqrt{3}}{2\sqrt{2}}} \frac{1}{u}\,du = 4\pi \log\left(\frac{2\sqrt{2}}{1+\sqrt{3}}\right)$$

③ La T være legemet i første oktant begrenset av den elliptiske sylinderen $x^2+2y^2=1$ og planet $z=1$



Beregn integralet $I = \iiint_T \frac{4xz}{\sqrt{x^2+y^2}}\,dx\,dy\,dz$

$$\frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos(\theta)}{r} = \cos(\theta)$$

I sylinderkoordinater

$$1 = x^2+2y^2 = \underbrace{r^2\cos(\theta)^2 + r^2\sin(\theta)^2}_{} + r^2\sin(\theta)^2 = r^2(1+\sin(\theta)^2)$$

(y^2 + y^2)

$$0 \le \theta \le \frac{\pi}{2} \qquad 0 \le r \le \frac{1}{\sqrt{1+\sin(\theta)^2}} \qquad 0 \le z \le 1$$

$$I = \int_0^{\pi/2}\int_0^{\frac{1}{\sqrt{1+\sin(\theta)^2}}}\int_0^1 4z\cos(\theta)\,\underbrace{r\,dz\,dr\,d\theta}_{dV} = \int_0^{\pi/2}\int_0^{\frac{1}{\sqrt{1+\sin(\theta)^2}}} 2r\cos(\theta)\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{\cos(\theta)}{1+\sin(\theta)^2}\,d\theta = \int_0^1 \frac{du}{1+u^2} = \arctan(u)\Big|_0^1 = \frac{\pi}{4}$$

$u = \sin(\theta) \quad du = \cos(\theta)\,d\theta$

Sylinder: Ex 2,3 p 869–870

Kule: Ex 4,5 p 871–873

Anm: Ex 1 p868 (Volum av ellipsoide)