Oppgave 1

(a) $f_x(x,y) = \dfrac{(x^2+y^2+4)\cdot 2(2x+3y)\cdot 2 - (2x+3y)^2\cdot 2x}{(x^2+y^2+4)^2}$

$$f_x(1,1) = \frac{6\cdot 2\cdot 5\cdot 2 - 5^2\cdot 2}{6^2} = \frac{120-50}{36} = \frac{70}{36} = \frac{35}{18}$$

$$f_y(x,y) = \frac{(x^2+y^2+4)\cdot 2(2x+3y)\cdot 3 - (2x+3y)^2\cdot 2y}{(x^2+y^2+4)^2}$$

$$f_y(1,1) = \frac{6\cdot 2\cdot 5\cdot 3 - 5^2\cdot 2}{6^2} = \frac{180-50}{36} = \frac{130}{36} = \frac{65}{18}$$

$$\nabla f(1,1) = \underline{\underline{\frac{35}{18}\vec{\imath} + \frac{65}{18}\vec{\jmath}}}\,.$$

En likning for tangenten i $(1,1)$ til nivåkurven til f gjennom $(1,1)$ er dermed:

$$\underline{\underline{\frac{35}{18}(x-1) + \frac{65}{18}(y-1) = 0}} \quad \text{dvs.:} \quad \underline{\underline{7x + 13y = 20}}\,.$$

(b) $D_{\vec{v}}f(1,1) = \dfrac{\nabla f(1,1)\cdot\vec{v}}{|\vec{v}|} = \dfrac{\nabla f(1,1)\cdot\vec{v}}{|\vec{v}|} = \dfrac{\frac{35}{18}\cdot 3 + \frac{65}{18}\cdot 4}{\sqrt{3^2+4^2}} = \underline{\underline{\dfrac{73}{18}}}\,.$

(c) Med $\Delta x = 0.1$ og $\Delta y = -0.1$ blir den lineære tilnærmineng til Δf lik

$$\Delta x\cdot f_x(1,1) + \Delta y\cdot f_y(1,1) = 0.1\cdot\frac{35}{18} - 0.1\cdot\frac{65}{18} = -\frac{3}{18} = \frac{1}{6} \approx \underline{\underline{-0.17}}\,.$$

(d) $f(1,1) = \dfrac{25}{6}$, så tangentplanet til grafen til f i det punktet på grafen som har $x = y = 1$ er gitt ved likningen

$$\underline{\underline{\frac{35}{18}\cdot(x-1) + \frac{65}{18}\cdot(y-1) - 1\cdot(z - \frac{25}{6}) = 0}} \quad \text{dvs.:} \quad \underline{\underline{35x + 65y - 18z = 25}}\,.$$

Oppgave 2

(a) Øvre halvdel av den første sirkelen, gitt ved $y = \sqrt{2x - x^2}$, skjærer nedre halvdel av c den andre, gitt ved $y = 1 - \sqrt{1-x^2}$, og de skjærer hverandre i $(0,0)$ og $(1,1)$, så vi har:

$$\iint_D dA = \underline{\underline{\int_0^1 \int_{1-\sqrt{1-x^2}}^{\sqrt{2x-x^2}} dy\,dx}}\,.$$

(b) De to sirklene er gitt ved $r = 2\cos\theta$ og $r = 2\sin\theta$, så vi har:

$$\iint_D dA = \underline{\underline{\int_0^{\pi/4}\int_0^{2\sin\theta} r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} r\,dr\,d\theta}}\,.$$

(c) Vi bruker polarkoordinater:

$$A_D = \iint_D dA = \int_0^{\pi/4}\int_0^{2\sin\theta} r\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} r\,dr\,d\theta =$$

$$= \int_0^{\pi/4} \left[\frac{1}{2}r^2\right]_0^{2\sin\theta} d\theta + \int_{\pi/4}^{\pi/2} \left[\frac{1}{2}r^2\right]_0^{2\cos\theta} d\theta =$$

$$= \int_0^{\pi/4} 2\sin^2\theta \, d\theta + \int_{\pi/4}^{\pi/2} 2\cos^2\theta \, d\theta =$$

$$= \int_0^{\pi/4} (1-\cos 2\theta)\, d\theta + \int_{\pi/4}^{\pi/2} (1+\cos 2\theta)\, d\theta =$$

$$= \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} + \left[\theta + \frac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/2} =$$

$$= (\frac{\pi}{4} - \frac{1}{2}) + ((\frac{\pi}{2} + 0) - (\frac{\pi}{4} + \frac{1}{2})) = \underline{\underline{\frac{\pi}{2} - 1}} \, .$$

Oppgave 3

Vektorfeltet $\vec{F}$ er definert i heler rommet $\mathbf{R}^3$ ved: $\vec{F}(x,y,z) = xz^2\,\vec{\imath} + x^2y\,\vec{\jmath} + y^2z\,\vec{k}$, og legemet T er den delen av av kula med sentrum i $(0,0,a)$ og radius a som er utenfor kula med sentrum i $(0,0,0)$ og radius a, dvs. T er gitt ved ulikhetene $x^2+y^2+(z-a)^2 \le a^2$ og $x^2+y^2+z^2 \ge a^2$ (a er en positiv konstant).

(a) $\nabla \cdot \vec{F} = z^2 + x^2 + y^2 = \underline{\underline{x^2+y^2+z^2}}$,

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz^2 & x^2y & y^2z \end{vmatrix} = \underline{\underline{2yz\,\vec{\imath} + 2xz\,\vec{\jmath} + 2xy\,\vec{k}}} \, .$$

(b) $V_T = \iiint_T dV = \int_0^{2\pi} \int_0^{\pi/3} \int_a^{2a\cos\phi} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta =$

$$= \int_0^{2\pi} \int_0^{\pi/3} (\frac{8}{3}a^3\cos^3\phi - \frac{1}{3}a^3) \sin\phi \, d\phi\, d\theta =$$

$$= \frac{8}{3}a^3 \int_0^{2\pi} d\theta \cdot \int_0^{\pi/3} \cos^3\phi \sin\phi\, d\phi - \frac{1}{3}a^3 \int_0^{2\pi} d\theta \cdot \int_0^{\pi/3} \sin\phi \, d\phi =$$

$$= \frac{8}{3}a^3 \cdot 2\pi \cdot \left[-\frac{1}{4}\cos^4\phi\right]_0^{\pi/3} - \frac{1}{3}a^3 \cdot 2\pi \cdot [-\cos\phi]_0^{\pi/3} =$$

$$= \frac{8}{3}a^3 \cdot 2\pi \cdot (-\frac{1}{4})((\frac{1}{2})^4 - 1) + \frac{1}{3}a^3 \cdot 2\pi \cdot (\frac{1}{2} - 1) =$$

$$= \frac{4}{3}\pi a^3 \cdot \frac{15}{16} - \frac{2}{3}\pi a^3 \cdot \frac{1}{2} = \underline{\underline{\frac{11}{12}\pi a^3}} \, .$$

(c) Med S lik hele overflaten til T, og $\vec{n}$ utadrettet fra T, får vi:

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_T \nabla\cdot\vec{F} dV = \iiint_T (x^2+y^2+z^2) dV = \int_0^{2\pi} \int_0^{\pi/3} \int_a^{2a\cos\phi} \rho^4 \sin\phi\, d\rho\, d\phi\, d\theta =$$

$$= \int_0^{2\pi} \int_0^{\pi/3} (\frac{32}{5}a^5\cos^5\phi - \frac{1}{5}a^5) \sin\phi\, d\phi\, d\theta =$$

$$= \frac{32}{5}a^5 \int_0^{2\pi} d\theta \cdot \int_0^{\pi/3} \cos^5\phi \sin\phi\, d\phi - \frac{1}{5}a^5 \int_0^{2\pi} d\theta \cdot \int_0^{\pi/3} \sin\phi\, d\phi =$$

$$= \frac{32}{5}a^5 \cdot 2\pi \cdot \left[-\frac{1}{6}\cos^6\phi\right]_0^{\pi/3} - \frac{1}{5}a^5 \cdot 2\pi \cdot [-\cos\phi]_0^{\pi/3} =$$

$$= \frac{32}{5}a^5 \cdot 2\pi \cdot (-\frac{1}{6})((\frac{1}{2})^6 - 1) + \frac{1}{5}a^5 \cdot 2\pi \cdot (\frac{1}{2} - 1) =$$

$$= \frac{32}{15}\pi a^5 \cdot \frac{63}{64} - \frac{2}{5}\pi a^5 \cdot \frac{1}{2} = \underline{\underline{\frac{19}{10}\pi a^5}}\ .$$

(d) Vi parametriserer S_1 med ϕ og θ som parametere, slik:

$$\vec{r}(\phi,\theta) = a\sin\phi\cos\theta\,\vec{\imath} + a\sin\phi\sin\theta\,jivec + a\cos\phi\,\vec{k}\ , \qquad 0 \le \phi \le \frac{\pi}{3}\ , \quad 0 \le \theta \le 2\pi\ ,$$

som gir:

$$\frac{\partial\vec{r}}{\partial\phi} \times \frac{\partial\vec{r}}{\partial\theta} = a^2(\sin^2\phi\cos\theta\,\vec{\imath} + \sin^2\phi\sin\theta\,\vec{\jmath} + \cos\phi\sin\phi\,\vec{k})$$

som er oppoverrettet (positiv 3.komponent), mens $\vec{n}$ skal være nedoverrettet, slik at

$$\iint_{S_1} \vec{F}\cdot\vec{n}\,dS = -\int_0^{2\pi}\int_0^{\pi/3} \vec{F}(\vec{r}(\phi,\theta)) \cdot \frac{\partial\vec{r}}{\partial\phi} \times \frac{\partial\vec{r}}{\partial\theta}\,d\phi\,d\theta =$$

$$= -\int_0^{2\pi}\int_0^{\pi/3} (a\sin\phi\cos\theta(a\cos\phi)^2\,\vec{\imath} + (a\sin\phi\cos\theta)^2 a\sin\phi\sin\theta\,\vec{\jmath} + (a\sin\phi\sin\theta)^2 a\cos\phi\,\vec{k}) \cdot$$
$$a^2(\sin^2\phi\cos\theta\,\vec{\imath} + \sin^2\phi\sin\theta\,\vec{\jmath} + \cos\phi\sin\phi\,\vec{k})\,d\phi\,d\theta =$$

$$= -a^5\int_0^{2\pi}\int_0^{\pi/3} (\sin^3\phi\cos^2\phi\cos^2\theta + \sin^5\phi\cos^2\theta\sin^2\theta + \sin^3\phi\cos^2\phi\sin^2\theta)\,d\phi\,d\theta =$$

$$= -a^5\int_0^{2\pi}\int_0^{\pi/3} (\sin^3\phi\cos^2\phi + \sin^5\phi\cos^2\theta\sin^2\theta)\,d\phi\,d\theta =$$

$$= -a^5(\int_0^{2\pi} d\theta \int_0^{\pi/3} \sin^3\phi\cos^2\phi\,d\phi + \int_0^{2\pi}\cos^2\theta\sin^2\theta\,d\theta\int_0^{\pi/3}\sin^5\phi\,d\phi) =$$

$$= -a^5(2\pi\int_0^{\pi/3}(1-\cos^2\phi)\cos^2\phi\sin\phi\,d\phi +$$
$$\int_0^{2\pi}\frac{1}{4}(1+\cos 2\theta)(1-\cos 2\theta)\,d\theta\int_0^{\pi/3}(1-\cos^2\phi)^2\sin\phi\,d\phi) =$$

$$= -a^5(-2\pi\int_1^{1/2}(1-u^2)u^2\,du - \frac{1}{4}\int_0^{2\pi}(1-\cos^2 2\theta)\,d\theta\int_1^{1/2}(1-u^2)^2\,du) =$$

$$= -a^5(-2\pi\int_1^{1/2}(u^2-u^4)\,du - \frac{1}{4}\int_0^{2\pi}(1-\frac{1}{2}(1+\cos 4\theta))\,d\theta\int_1^{1/2}(1-2u^2+u^4)\,du) =$$

$$= -a^5(-2\pi\left[\frac{u^3}{3}-\frac{u^5}{5}\right]_1^{1/2} - \frac{1}{8}\int_0^{2\pi}(1-\cos 4\theta)\,d\theta\left[u - \frac{2u^3}{3} + \frac{u^5}{5}\right]_1^{1/2}) =$$

$$= -a^5(-2\pi((\frac{1}{24}-\frac{1}{160}) - (\frac{1}{3}-\frac{1}{5})) -$$
$$\frac{1}{8}(\int_0^{2\pi}d\theta - \int_0^{2\pi}\cos 4\theta)\,d\theta)((\frac{1}{2}-\frac{1}{12}+\frac{1}{160}) - (1-\frac{2}{3}+\frac{1}{5}))) =$$

$$= \cdots = \underline{\underline{-\frac{427}{1920}\pi a^5}}\ .$$

$$\iint_{S_2} \vec{F}\cdot\vec{n}\,dS = \iint_{S} \vec{F}\cdot\vec{n}\,dS - \iint_{S_1} \vec{F}\cdot\vec{n}\,dS = \frac{19}{10}\pi a^5 - (-\frac{427}{1920}\pi a^5) = \underline{\underline{\frac{815}{384}\pi a^5}}\ .$$

Oppgave 4

(a) Neste side.

(b) $E_n = \frac{2}{20}\int_0^{20} u(x,0)\sin\frac{n\pi x}{20}dx = \frac{1}{10}(\int_0^2 x\sin\frac{n\pi x}{20}dx + \int_2^4 (4-x)\sin\frac{n\pi x}{20}dx + 0) =$

$$= \frac{1}{10}(\Big[-\frac{20}{n\pi}x\cos\frac{n\pi x}{20} + (\frac{20}{n\pi})^2\sin\frac{n\pi x}{20}\Big]_0^2 + \Big[-\frac{20}{n\pi}(4-x)\cos\frac{n\pi x}{20} - (\frac{20}{n\pi})^2\sin\frac{n\pi x}{20}\Big]_2^4) =$$

$$= \frac{1}{10}(-\frac{40}{n\pi}\cos\frac{n\pi}{10} + (\frac{20}{n\pi})^2\sin\frac{n\pi}{10} - (\frac{20}{n\pi})^2\sin\frac{n\pi}{5} + \frac{40}{n\pi}\cos\frac{n\pi}{10} + (\frac{20}{n\pi})^2\sin\frac{n\pi}{10}) =$$

$$= \underline{\underline{\frac{40}{n^2\pi^2}(2\sin\frac{n\pi}{10} - \sin\frac{n\pi}{5})}}\ .$$

(a) $u(x,t)$ er løsningen av en bølgelikning med $c = 1$, $L = 20$ og $f(x) = u(x,0)$, og iflg. d'Alemberts løsning er da $u(x,1) = \frac{1}{2}(\tilde{f}(x+t)+\tilde{f}(x-t))$, $u(x,2) = \frac{1}{2}(\tilde{f}(x+2)+\tilde{f}(x-2))$, $u(x,3) = \frac{1}{2}(\tilde{f}(x+3)+\tilde{f}(x-3))$ og $u(x,4) = \frac{1}{2}(\tilde{f}(x+4)+\tilde{f}(x-4))$, der $\tilde{f}$ er den odde halvperiodiske utvidelsen av f med periode 40. Grafene til disse, samt startsituasjonen med $f(x) = u(x,0)$, for $0 \le x \le 8$, ser slik ut: