NTNU I GJØVIK

LØSNINGER, EKSAMEN I MATEMATIKK 3, 7. APRIL 2017

Oppgave 1

(a) En normalvektor til S i P:

$$-\frac{\partial f}{\partial x}\vec{\imath}-\frac{\partial f}{\partial y}\vec{\jmath}+\vec{k}=-\frac{(x^2+2y^2)\cdot y^2-xy^2\cdot 2x}{(x^2+2y^2)^2}\vec{\imath}-\frac{(x^2+2y^2)\cdot 2xy-xy^2\cdot 4y}{(x^2+2y^2)^2}\vec{\jmath}+\vec{k}=$$

$$=-\frac{y^2(2y^2-x^2)}{(x^2+2y^2)^2}\vec{\imath}-\frac{2x^3y}{(x^2+2y^2)^2}\vec{\jmath}+\vec{k}=\underline{\underline{-\frac{28}{81}\vec{\imath}-\frac{4}{81}\vec{\jmath}+\vec{k}}}$$

(b) En likning for tangentplanet til S i P.

$$\underline{\underline{-\frac{28}{81}(x-1)-\frac{4}{81}(y-2)+z-\frac{4}{9}=0}} \quad \text{dvs.:} \quad \underline{\underline{z=\frac{4}{81}(7x+y)}}$$

(c) Lineærapproksimasjonen til endringen Δf i $f(x,y,z)$ ved flytting fra $(x,y)=(1,2)$ til $(x,y)=(1+\Delta x, 2+\Delta y)$ er lik Δz for tangentplanfunksjonen:

$$\Delta z=\frac{4}{81}(7(1+\Delta x)+(2+\Delta y))-\frac{4}{81}(7\cdot 1+2)=\underline{\underline{\frac{4}{81}(7\Delta x+\Delta y)}}$$

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Oppgave 2

De to sirklene skjærer hverandre i origo og i et punkt til på linja $y=x$, så integralet må settes opp som en sum av to deler, en del for $0\le\theta\le\pi/4$ og en annen del for $\pi/4\le\theta\le\pi/2$. Sirkellikningene i polarkoordinater er: $(x-1)^2+y^2=1 \Leftrightarrow r=2\cos\theta$, og $x^2+(y-1)^2=1 \Leftrightarrow r=2\sin\theta$. Dermed får vi:

$$\iint_D (x^2+y^2)dA=\underline{\underline{\int_0^{\pi/4}\int_0^{2\sin\theta} r^3\,dr\,d\theta+\int_{\pi/4}^{\pi/2}\int_0^{2\cos\theta} r^3\,dr\,d\theta}}$$

som videre er lik:

$$\int_0^{\pi/4}\left[\frac{1}{4}r^4\right]_0^{2\sin\theta}d\theta+\int_{\pi/4}^{\pi/2}\left[\frac{1}{4}r^4\right]_0^{2\cos\theta}d\theta=$$

$$=\frac{1}{4}\left(\int_0^{\pi/4}16\sin^4\theta\,d\theta+\int_{\pi/4}^{\pi/2}16\cos^4\theta\,d\theta\right)=$$

$$=4\left(\int_0^{\pi/4}\sin^4\theta\,d\theta+\int_{\pi/4}^{\pi/2}\cos^4\theta\,d\theta\right)=\cdots=$$

$$=4\left(\int_0^{\pi/4}\frac{1}{8}(3-4\cos 2\theta+\cos 4\theta)\,d\theta+\int_{\pi/4}^{\pi/2}\frac{1}{8}(3+4\cos 2\theta+\cos 4\theta)\,d\theta\right)=$$

$$= \frac{1}{2}\left(\left[3\theta - 2\sin 2\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} + \left[3\theta + 2\sin 2\theta + \frac{1}{4}\sin 4\theta\right]_{\pi/4}^{\pi/2}\right) = \cdots = \underline{\underline{\frac{3}{4}\pi - 2}}.$$

Oppgave 3

Vi finner først kulelikningen i kulekoordinater: $(x-1)^2+y^2+z^2 = 1 \Leftrightarrow \rho = 2\sin\phi\cos\theta$. Kula ligger på den "$x$-positive" siden av yz-planet, så θ må begrenses til $-\pi/2 \leq \theta \leq \pi/2$ (evt. $0 \leq \theta \leq \pi/2$ og $3\pi/2 \leq \theta \leq 2\pi$), og vi får:

$$\iiint_T (x^2 + y^2 + z^2)dV = \underline{\underline{\int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^{2\sin\phi\cos\theta} \rho^4 \sin\phi \, d\rho \, d\phi \, d\theta.}}$$

Oppgave 4

$\underline{\underline{\text{Nei}}}$, det fins ingen slik funksjon f, fordi feltet er ikke curl-fritt, og dermed heller ikke konservativt i $\mathbf{R}^3$:

$$\nabla \times (xy\,\vec{\imath} + yz\,\vec{\jmath} + xz\,\vec{k}) = \cdots = -y\,\vec{\imath} - z\,\vec{\jmath} - x\,\vec{k}.$$

Oppgave 5

(a) $\quad \text{curl}(\vec{F}) = \nabla \times (xy\,\vec{\imath} + yz\,\vec{\jmath} + xz\,\vec{k}) = \cdots = \underline{\underline{-y\,\vec{\imath} - z\,\vec{\jmath} - x\,\vec{k}}},$
og $\text{div}(\vec{F}).= \nabla \cdot (xy\,\vec{\imath} + yz\,\vec{\jmath} + xz\,\vec{k}) = \underline{\underline{y + z + x}}.$

(b) $\quad \int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^2 \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t)\,dt =$

$$= \int_0^2 (t^2 \cdot t^3\,\vec{\imath} + t^3 \cdot t^4\,\vec{\jmath} + t^2 \cdot t^4\,\vec{k}) \cdot (2t\vec{\imath} + 3t^2\vec{\jmath} + 4t^3\vec{k})\,dt =$$

$$= \int_0^2 (2t^6 + 3t^9 + 4t^9)\,dt = \int_0^2 (2t^6 + 7t^9)\,dt = \frac{2 \cdot 2^7}{7} + \frac{7 \cdot 2^{10}}{10} = \underline{\underline{\frac{26368}{35}}}.$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^4 \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t)\,dt =$$

$$= \int_0^4 ((4-t)(8-2t)\,\vec{\imath} + (8-2t)(16-4t)\,\vec{\jmath} + (4-t)(16-4t)\,\vec{k}) \cdot (-\vec{\imath} - 2\vec{\jmath} - 4\vec{k})\,dt =$$

$$= \int_0^4 (-(32 - 16t + 2t^2) - 2(128 - 64t + 8t^2) - 4(64 - 32t + 4t^2))\,dt =$$

$$= -17\int_0^4 (32 - 16t + 2t^2)dt = -17 \cdot (32 \cdot 4 - 8 \cdot 4^2 + \frac{2}{3} \cdot 4^3) = \underline{\underline{-\frac{2176}{3}}}.$$

(c) $\quad \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \frac{26368}{35} - \frac{2176}{3} = \cdots = \underline{\underline{\frac{2944}{105}}}.$

Oppgave 6

(a) S_1 er grafen til $z = 2x+3y$, $x^2+y^2 \le 3$, så vi parametriserer den med $\vec{r}(x,y) = x\vec{\imath} + y\vec{\jmath} + (2x+3y)\vec{k}$, som gir $\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = -2\vec{\imath} - 3\vec{\jmath} + \vec{k}$, og dermed:

$$\iint_{S_1} \vec{F}\cdot\vec{n}\,dS = \iint_{(S_1)_{xy}} \vec{F}(x,y,2x+3y)\cdot\frac{\partial \vec{r}}{\partial x}\times\frac{\partial \vec{r}}{\partial y}\,dx\,dy =$$

$$= \iint_{(S_1)_{xy}} (2xy\,\vec{\imath} - y^2\,\vec{\jmath})\cdot(-2\vec{\imath} - 3\vec{\jmath} + \vec{k})\,dx\,dy =$$

$$= \iint_{(S_1)_{xy}} (-4xy+3y^2)\,dx\,dy = \int_0^{2\pi}\int_0^3 (-4r^2\cos\theta\sin\theta + 3r^2\sin^2\theta)\,r\,dr\,d\theta =$$

$$= -4\int_0^{2\pi}\cos\theta\sin\theta\,d\theta\cdot\int_0^3 r^3 dr + 3\int_0^{2\pi}\sin^2\theta\,d\theta\cdot\int_0^3 r^3 dr =$$

$$= -4\int_0^{2\pi}\frac{1}{2}\sin 2\theta\,d\theta\cdot\frac{3^4}{4} + 3\int_0^{2\pi}\frac{1}{2}(1-\cos 2\theta)\,d\theta\cdot\frac{3^4}{4} =$$

$$= 0 + 3\cdot\frac{81}{4}\left[\frac{1}{2}(\theta - \frac{1}{2}\sin 2\theta)\right]_0^{2\pi} = \underline{\underline{\frac{243}{4}\pi}}.$$

(b) $\vec{F}$ er divergensfritt i hele $\mathbf{R}^3$, så iflg. Divergensteoremet er den samlede fluksen ut fra T 0, slik at

$$\iint_{S_2} \vec{F}\cdot\vec{n}\,dS = 0 - \iint_{S_1} \vec{F}\cdot\vec{n}\,dS = \underline{\underline{-\frac{243}{4}\pi}}.$$