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LØSNINGER, EKSAMEN I MATEMATIKK 3

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Oppgave 1

(a) $\nabla f(x,y,z) = (2xy+z^2)\vec{\imath} + (x^2+2yz)\vec{\jmath} + (y^2+2xz)\vec{k}$, så
$\nabla f(P) = (2\cdot 3\cdot 2+1^2)\vec{\imath} + (3^2+2\cdot 2\cdot 1)\vec{\jmath} + (2^2+2\cdot 3\cdot 1)\vec{k} = 13\vec{\imath}+13\vec{\jmath}+10\vec{k}$,
så en likning for tangentplanet til S i P er:
$\underline{\underline{13(x-3)+13(y-2)+10(z-1)=0}}$, dvs.: $\underline{\underline{13x+13y+10z=75}}$.

(b) Enhetsvektoren som peker i retningen for sterkest mulig økning av f-verdier i P er:
$$\frac{\nabla f(P)}{|\nabla f(P)|} = \frac{13\vec{\imath}+13\vec{\jmath}+10\vec{k}}{\sqrt{13^2+13^2+10^2}} = \underline{\underline{\frac{1}{\sqrt{438}}(13\vec{\imath}+13\vec{\jmath}+10\vec{k})}}.$$

Oppgave 2

(a) $$\iint_R f(x,y)dA = \underline{\underline{\int_0^1 \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} \sqrt{x^2+y^2}\,dx\,dy}}$$

(b) $$\iint_R f(x,y)dA = \underline{\underline{\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx}}$$

(c) $$\iint_R f(x,y)dA = \underline{\underline{\int_0^{\pi/2} \int_0^{2\cos\theta} r^2\,dr\,d\theta}}$$

(d) $$\iint_R f(x,y)dA = \int_0^{\pi/2} \int_0^{2\cos\theta} r^2\,dr\,d\theta = \int_0^{\pi/2} \frac{8}{3}\cos^3\theta\,d\theta = \frac{8}{3}\int_0^{\pi/2}(1-\sin^2\theta)\cos\theta\,d\theta =$$
$$= \frac{8}{3}\int_0^1 (1-u^2)\,du = \frac{8}{3}(1-\frac{1}{3}) = \underline{\underline{\frac{16}{9}}}$$

Oppgave 3

$$\iiint_T x^2 dV = \int_0^2 \int_0^{x/2} \int_0^y x^2\,dz\,dy\,dx + \int_0^2 \int_{x/2}^x \int_0^{x-y} x^2\,dz\,dy\,dx =$$
$$= \int_0^2 \int_0^{x/2} x^2 y\,dy\,dx + \int_0^2 \int_{x/2}^x x^2(x-y)\,dy\,dx =$$
$$= \int_0^2 x^2\cdot\frac{1}{2}(\frac{x}{2})^2\,dx + \int_0^2 [x^3y - \frac{1}{2}x^2y^2]_{y=x/2}^{y=x}\,dx =$$
$$= \frac{1}{8}\int_0^2 x^4\,dx + \frac{1}{8}\int_0^2 x^4\,dx = \frac{1}{4}\cdot\frac{2^5}{5} = \underline{\underline{\frac{8}{5}}}$$

Oppgave 4

(a) $$L_C = \int_0^1 |\vec{r}\,'(t)|dt = \int_0^1 |-6\pi\sin(2\pi t)\,\vec{\imath} + 6\pi\cos(2\pi t)\,\vec{\jmath} + 4\,\vec{k}|dt =$$
$$= \int_0^1 \sqrt{36\pi^2(\sin^2(2\pi t)+\cos^2(2\pi t))+16}\,dt = \int_0^1 \sqrt{36\pi^2+16}\,dt = \underline{\underline{2\sqrt{9\pi^2+4}}}$$

(b) $\int_C (2y\vec{\imath} + 3z\vec{\jmath}) \cdot d\vec{r} = \int_0^1 (6\sin(2\pi t)\,\vec{\imath} + 12t\,\vec{\jmath}) \cdot \vec{r}\,'(t)\,dt =$

$$= \int_0^1 (6\sin(2\pi t)\,\vec{\imath} + 12t\,\vec{\jmath}) \cdot (-6\pi\sin(2\pi t)\,\vec{\imath} + 6\pi\cos(2\pi t)\,\vec{\jmath} + 4\,\vec{k})\,dt =$$

$$= \int_0^1 (-36\sin^2(2\pi t) + 72\pi t\cos(2\pi t) + 0)\,dt =$$

$$= -36\int_0^1 \sin^2(2\pi t)\,dt + 72\pi\int_0^1 t\cos(2\pi t)\,dt =$$

$$= -36[\frac{1}{2}(t - \frac{1}{4\pi}\sin(4\pi t)]_0^1 + 72\pi[t\cdot\frac{1}{2\pi}\sin(2\pi t) - \int\frac{1}{2\pi}\sin(2\pi t)\,dt]_0^1 =$$

$$= -18 + 72\pi[t\cdot\frac{1}{2\pi}\sin(2\pi t) + \frac{1}{(2\pi)^2}\cos(2\pi t)]_0^1 =$$

$$= -18 + 72\pi((0 + \frac{1}{(2\pi)^2}) - (0 + \frac{1}{(2\pi)^2})) = \underline{\underline{-18}}$$

Oppgave 5

(a) $\nabla\cdot\vec{F} = \frac{\partial z^2}{\partial x} + \frac{\partial(-x^2)}{\partial y} + \frac{\partial y^2}{\partial z} = \underline{\underline{0}}$

$$\nabla\times\vec{F} = (\frac{\partial y^2}{\partial y} - \frac{\partial(-x^2)}{\partial z})\,\vec{\imath} - (\frac{\partial y^2}{\partial x} - \frac{\partial(z^2)}{\partial z})\,\vec{\jmath} + (\frac{\partial(-x^2)}{\partial x} - \frac{\partial z^2}{\partial y})\,\vec{k} = \underline{\underline{2y\,\vec{\imath} + 2z\,\vec{\jmath} - 2x\,\vec{k}}}$$

$\vec{F}$ er <u>ikke konservativt</u> i $\mathbf{R}^3$ da $\nabla\times\vec{F}$ ikke er konstant lik $\vec{0}$.

(b) Vi beregner $\iint_S \vec{F}\cdot\vec{n}\,dS$ ved hjelp av Divergensteoremet:

Flaten S er en del av overflaten til den 4.dels-kula T som har sentrum i origo, radius $\sqrt{5}$, og ligger i den z-positive og x-positive delen av $\mathbf{R}^3$. Resten av overflaten til T er et halvsirkelområde D_1 i xy-planet (med utadrettet enhetsnormalvektor $-\vec{k}$) og et halvsirkelområde D_2 i yz-planet (med utadrettet enhetsnormalvektor $-\vec{\imath}$). Divergensteoremet sier da at

$$\iiint_T \nabla\cdot\vec{F}\,dV = 0 = \iint_S \vec{F}\cdot\vec{n}\,dS + \iint_{D_1}\vec{F}\cdot(-\vec{k})\,dS + \iint_{D_2}\vec{F}\cdot(-\vec{\imath})\,dS$$

slik at $\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_{D_1}\vec{F}\cdot\vec{k}\,dS + \iint_{D_2}\vec{F}\cdot\vec{\imath}\,dS = \iint_{D_1} y^2\,dS + \iint_{D_2} z^2\,dS =$

(bruker polarkoordinater i både xy-planet og yz-planet)

$$= \int_{-\pi/2}^{\pi/2}\int_0^{\sqrt{5}} (r\sin\theta)^2\cdot r\,dr\,d\theta + \int_{-\pi/2}^{\pi/2}\int_0^{\sqrt{5}} (r\sin\theta)^2\cdot r\,dr\,d\theta =$$

$$= 2\int_0^{\sqrt{5}} r^3 dr\cdot\int_{-\pi/2}^{\pi/2}\sin^2\theta\,d\theta = 2\cdot\frac{1}{4}(\sqrt{5})^4\cdot[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta]_{-\pi/2}^{\pi/2} = \underline{\underline{25\pi}}$$

(c) Vi beregner $\iint_S \nabla\times\vec{F}\cdot\vec{n}\,dS$ ved hjelp av Stokes' teorem:

Randkurven C til flaten S består av en halvsirkelbue C_1 i den x-positive delen av xy-planet (som her skal være orientert fra $(0, -\sqrt{5}, 0)$ til $(0, \sqrt{5}, 0)$) og en halvsirkelbue C_2 i den z-positive delen av yz-planet (som her skal være orientert fra $(0, \sqrt{5}, 0)$ til $(0, -\sqrt{5}, 0)$). Stokes' teorem sier da at

$$\iint_S \nabla \times \vec{F} \cdot \vec{n}\, dS = \oint_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} =$$

(bruker parametriseringene $C_1 : \vec{r}_1(t) = \sqrt{5} \cos t\, \vec{\imath} + \sqrt{5} \sin t\, \vec{\jmath}$ og

$C_2 : \vec{r}_2(t) = -\sqrt{5} \sin t\, \vec{\jmath} + \sqrt{5} \cos t \vec{k}$, begge med $t : -\pi/2 \to \pi/2$)

$$= \int_{-\pi/2}^{\pi/2} \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t) dt + \int_{-\pi/2}^{\pi/2} \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t) dt =$$

$$= \int_{-\pi/2}^{\pi/2} (-5 \cos^2 t \vec{\jmath} + 5 \sin^2 t \vec{k}) \cdot \sqrt{5}(-\sin t \vec{\imath} + \cos t \vec{\jmath}) dt +$$

$$\int_{-\pi/2}^{\pi/2} (5 \cos^2 t \vec{\imath} + 5 \sin^2 t \vec{k}) \cdot \sqrt{5}(-\cos t \vec{\jmath} - \sin t \vec{k}) dt =$$

$$= 5\sqrt{5} \int_{-\pi/2}^{\pi/2} (-\cos^3 t - \sin^3 t) dt = -5\sqrt{5} \int_{-\pi/2}^{\pi/2} (\cos^3 t + \sin^3 t) dt =$$

$$= -\frac{5\sqrt{5}}{3} [(2 + \cos^2 t) \sin t - (2 + \sin^2 t) \cos t]_{-\pi/2}^{\pi/2} =$$

$$= -\frac{5\sqrt{5}}{3} ((2 - 0) - (-2 - 0)) = \underline{\underline{-\frac{20\sqrt{5}}{3}}}$$