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Oppgave 1

(a) $\nabla f(x,y) = \dfrac{\partial f}{\partial x}\,\vec{\imath} + \dfrac{\partial f}{\partial y}\,\vec{\jmath} = \pi\cos\pi x\,\sin\pi y\,\vec{\imath} + \pi\sin\pi x\,\cos\pi y\,\vec{\jmath}$, slik at

$$\nabla f(1/3,2/3) = \pi\cos\frac{\pi}{3}\,\sin\frac{2\pi}{3}\,\vec{\imath} + \pi\sin\frac{\pi}{3}\,\cos\frac{2\pi}{3}\,\vec{\jmath} =$$

$$= \pi\cdot\frac{1}{2}\cdot\frac{\sqrt{3}}{2}\,\vec{\imath} + \pi\cdot\frac{\sqrt{3}}{2}\cdot(-\frac{1}{2})\,\vec{\imath} = \underline{\underline{\frac{\sqrt{3}\,\pi}{4}\,(\vec{\imath}-\vec{\jmath})}}\,.$$

(b) $f(1/3,2/3) = \sin\dfrac{\pi}{3}\,\sin\dfrac{2\pi}{3} = \dfrac{\sqrt{3}}{2}\cdot\dfrac{\sqrt{3}}{2} = \dfrac{3}{4}$, så grafen til likningen

$\underline{\underline{z = \dfrac{\sqrt{3}\,\pi}{4}(x-\dfrac{1}{3}) - \dfrac{\sqrt{3}\,\pi}{4}(y-\dfrac{2}{3}) + \dfrac{3}{4}}}$, eller $\underline{\underline{\sqrt{3}\,\pi(x-y) - 4z + \dfrac{\sqrt{3}\,\pi}{3} + 3}}$ er tangent-planet til grafen til f i punktet $(1/3, 2/3, f(1/3,2/3))$..

(c) Den maksimale verdien for den retningsderiverte til f i P er

$$|\nabla f(1/3,2/3)| = |\frac{\sqrt{3}\,\pi}{4}\,(\vec{\imath}-\vec{\jmath})| = \frac{\sqrt{3}\,\pi}{4}\,|\vec{\imath}-\vec{\jmath}| = \frac{\sqrt{3}\,\pi}{4}\cdot\sqrt{1+1} = \underline{\underline{\frac{\sqrt{6}\,\pi}{4}}}.$$

(d) $V = \displaystyle\int_0^1\int_0^1 \sin\pi x\,\sin\pi y\,dy\,dx = \int_0^1 \sin\pi x\,dx\cdot\int_0^1\sin\pi y\,dy =$

$$= \left[\frac{1}{\pi}(-\cos\pi x)\right]_0^1\cdot\left[\frac{1}{\pi}(-\cos\pi y)\right]_0^1 = (\frac{1}{\pi}\cdot 2)^2 = \underline{\underline{\frac{4}{\pi^2}}}\,.$$

Oppgave 2

Hjørnene P_1 og P_2 har samme xy-projeksjon, så T_{xy} (xy-projeksjonen ac T) er trekanten med hjørner $(1,2)$, $(2,0)$ og $(0,0)$.

Planet gjennom P_2, P_3 og P_4 er grafen til $z = \frac{3}{2}x - \frac{3}{4}y$, og planet gjennom P_1, P_3 og P_4 er grafen til $z = \frac{3}{2}x + \frac{3}{4}y$. Dermed er volumet av T:

$$V_T = \iiint_T dV = \iint_{T_{xy}}\int_{\frac{3}{2}x-\frac{3}{4}y}^{\frac{3}{2}x+\frac{3}{4}y} dz\,dA = \iint_{T_{xy}}((\frac{3}{2}x+\frac{3}{4}y)-(\frac{3}{2}x-\frac{3}{4}y))dA = \iint_{T_{xy}}(\frac{3}{2}y)dA =$$

$$= \frac{3}{2}\int_0^2\int_{y/2}^{2-y/2} y\,dx\,dy = \frac{3}{2}\int_0^2 y((2-y/2)-(y/2))\,dy = \frac{3}{2}\int_0^2(2y-y^2)dy = \underline{\underline{2}}\,.$$

Oppgave 3

(a) Med T = jordkloden har vi:

$$M = \iiint_T \delta(\rho)\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (a-b\rho)\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta =$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\pi} \sin\phi\, d\phi \cdot \int_0^{R} (a-b\rho)\rho^2\, d\rho = 2\pi \cdot 2 \cdot \int_0^{R} (a\rho^2 - b\rho^3)\, d\rho =$$

$$= 4\pi(\frac{1}{3}aR^3 - \frac{1}{4}bR^4) = \underline{\underline{\frac{\pi}{3}R^3(4a-3bR)}}\,.$$

(b) Det er oppgitt at $\delta(R) = a - bR = 3{,}000\,\mathrm{g/cm}^3$, slik at $a = 3{,}000\,\mathrm{g/cm}^3 + bR$, som gir at

$$M = \frac{\pi}{3}R^3(4a-3bR) = \frac{\pi}{3}R^3(4(3{,}000\,\mathrm{g/cm}^3 + bR) - 3bR) = \frac{\pi}{3}R^3(bR + 12{,}000\,\mathrm{g/cm}^3)\,.$$

slik at $Rb + 12{,}000\,\mathrm{g/cm}^3 = M/(\frac{\pi}{3}R^3)$, og dermed

$$bR = M/(\frac{\pi}{3}R^3) - 12{,}000\,\mathrm{g/cm}^3 = \frac{3M}{\pi R^3} - 12{,}000\,\mathrm{g/cm}^3.$$

Vi får da at

$$\delta(0) = a + b\cdot 0 = a = 3{,}000\,\mathrm{g/cm}^3 + bR = 3{,}000\,\mathrm{g/cm}^3 + \frac{3M}{\pi R^3} - 12{,}000\,\mathrm{g/cm}^3 =$$

$$= \frac{3M}{\pi R^3} - 9{,}000\,\mathrm{g/cm}^3 = \frac{3\cdot 5{,}972\cdot 10^{27}\mathrm{g}}{\pi(6{,}371\cdot 10^8)^3\mathrm{cm}^3} - 9{,}000\mathrm{g/cm}^3 =$$

$$= (\frac{3\cdot 5{,}972\cdot 10^3}{\pi(6{,}371)^3} - 9{,}000)\mathrm{g/cm}^3 = \underline{\underline{13{,}05\,\mathrm{g/cm}^3}}\,.$$

Oppgave 4

(a) $\mathrm{div}\vec{F} = \underline{\underline{2z+2x+2y}} = \underline{\underline{2(x+y+z)}}$

$\mathrm{curl}\vec{F} = (2(z-x)-(2z-2x))\,\vec{\imath} - ((2x-2y)-2(x-y))\,\vec{\jmath} + (2(y-z)-(2y-2z))\,\vec{k} = \underline{\underline{\vec{0}}}.$

(b) $\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_T \mathrm{div}\vec{F}\, dV = \iiint_T (2z+2x+2y)\, dV =$

$$= 2\int_0^{\pi/2}\int_0^{\pi/3}\int_2^{3} (\rho\cos\phi + \rho\sin\phi\cos\theta + \rho\sin\phi\sin\theta)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta =$$

$$= 2(\int_0^{\pi/2}\int_0^{\pi/3}\int_2^{3} \rho^3\cos\phi\sin\phi\, d\rho\, d\phi\, d\theta +$$

$$+ \int_0^{\pi/2}\int_0^{\pi/3}\int_2^{3} \rho^3\sin^2\phi\cos\theta\, d\rho\, d\phi\, d\theta +$$

$$+ \int_0^{\pi/2}\int_0^{\pi/3}\int_2^{3} \rho^3\sin^2\phi\sin\theta\, d\rho\, d\phi\, d\theta) =$$

$$= 2\Big(\int_0^{\pi/2} d\theta \cdot \int_0^{\pi/3} \cos\phi \sin\phi \, d\phi \cdot \int_2^3 \rho^3 \, d\rho +$$

$$+ \int_0^{\pi/2} \cos\theta \, d\theta \cdot \int_0^{\pi/3} \sin^2\phi \, d\phi \cdot \int_2^3 \rho^3 \, d\rho =$$

$$+ \int_0^{\pi/2} \sin\theta \, d\theta \cdot \int_0^{\pi/3} \sin^2\phi \, d\phi \cdot \int_2^3 \rho^3 \, d\rho\Big) =$$

$$= 2\Big(\frac{\pi}{2} \cdot \left[\frac{1}{2}\sin^2\phi\right]_0^{\pi/3} \cdot \left[\frac{1}{4}\rho^4\right]_2^3 + [\sin\theta]_0^{\pi/2} \cdot \left[\frac{1}{2}(\phi - \frac{1}{2}\sin 2\phi)\right]_0^{\pi/3} \cdot \left[\frac{1}{4}\rho^4\right]_2^3 +$$

$$+ [-\cos\theta]_0^{\pi/2} \cdot \left[\frac{1}{2}(\phi - \frac{1}{2}\sin 2\phi)\right]_0^{\pi/3} \cdot \left[\frac{1}{4}\rho^4\right]_2^3\Big) =$$

$$= 2\Big(\frac{\pi}{2} \cdot \frac{1}{2}(\frac{\sqrt{3}}{2})^2 \cdot \frac{1}{4}(3^4 - 2^4) + 1 \cdot \frac{1}{2}(\frac{\pi}{3} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2}) \cdot \frac{1}{4}(3^4 - 2^4) +$$

$$+ 1 \cdot \frac{1}{2}(\frac{\pi}{3} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2}) \cdot \frac{1}{4}(3^4 - 2^4) =$$

$$= 2\Big(\frac{\pi}{2} \cdot \frac{1}{2}(\frac{\sqrt{3}}{2})^2 \cdot \frac{1}{4}(3^4 - 2^4) + 1 \cdot \frac{1}{2}(\frac{\pi}{3} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2}) \cdot \frac{1}{4}(3^4 - 2^4) +$$

$$+ 1 \cdot \frac{1}{2}(\frac{\pi}{3} - \frac{1}{2} \cdot \frac{\sqrt{3}}{2}) \cdot \frac{1}{4}(3^4 - 2^4) =$$

$$= 2\Big(\frac{3\pi}{16} \cdot \frac{65}{4} + \frac{1}{2}(\frac{\pi}{3} - \frac{\sqrt{3}}{4}) \cdot \frac{65}{4} + \frac{1}{2}(\frac{\pi}{3} - \frac{\sqrt{3}}{4}) \cdot \frac{65}{4}\Big) = 2\Big(\frac{3\pi}{16} + \frac{\pi}{3} - \frac{\sqrt{3}}{4}\Big) \cdot \frac{65}{4} = \underline{\underline{\frac{1625}{96}\pi - \frac{65}{8}\sqrt{3}}}$$

(c) $\oint_C \vec{F} \cdot d\vec{r} = \underline{\underline{0}}$, fordi C er en lukket kurve, og $\vec{F}$ er konservativt i hele $\mathbf{R}^3$, i og med at curl$\vec{F} = 0$ og $\mathbf{R}^3$ er enkeltsammenhengende.

Oppgave 5

$u(x.0) = 20x$, fordi dette er den stabile temperaturfunksjonen med verdi 0 ved $x = 0$ og 20 ved $x = 1$.

For $t > 0$ skal dermed $u(x,t)$ oppfylle disse kravene:

(1) $u_t = \alpha^2 u_{xx}$, (2) $u(0,t) = -10$ $(t > 0)$, (3) $u(1,t) = 20$ $(t > 0)$
og (4) $u(x,0) = 20x$ $(0 \leq x \leq 1)$.

Vi setter nå $v(x,t) = u(x,t) - (30x - 10)$. Denne vil oppfylle disse kravene:

(1′) $v_t = \alpha^2 v_{xx}$, (2′) $v(0,t) = 0$ $(t > 0)$, (3′) $v(1,t) = 0$ $(t > 0)$
og (4′) $v(x,0) = 10 - 10x$ $(0 \leq x \leq 1)$

hvis og bare hvis $u(x,t)$ oppfyller (1) – (4).

Vi vet at (1′) – (4′) har løsningen $v(x,t) = \sum_{n=1}^{\infty} E_n \sin n\pi x \, e^{-n^2\pi^2\alpha^2 t}$ med

$$E_n = 2\int_0^1 (10 - 10x)\sin n\pi x \, dx = 20\left[(1-x) \cdot \frac{-1}{n\pi}\cos n\pi x - \int (-1)\frac{-1}{n\pi}\cos n\pi x \, dx\right]_0^1 =$$

$$= 20\left[(1-x)\cdot\frac{-1}{n\pi}\cos n\pi x - \frac{1}{n^2\pi^2}\sin n\pi x\right]_0^1 = \frac{20}{n\pi}\ .$$

Dermed har vi:

$$u(x,t) = 30x - 10 + v(x,t) = 30x - 10 + \sum_{n=1}^{\infty} E_n \sin n\pi x\, e^{-n^2\pi^2\alpha^2 t} =$$

$$= \underline{\underline{30x - 10 + \sum_{n=1}^{\infty} \frac{20}{n\pi}\sin n\pi x\, e^{-n^2\pi^2\alpha^2 t}}}\ .$$