HØGSKOLEN I GJØVIK

LØSNINGER, EKSAMEN I MATEMATIKK 3 20.12.2013

Oppgave 1

(a) $\nabla f(P) = f_x(P)\vec{\imath} + f_y(P)\vec{\jmath} + f_z(P)\vec{k} =$

$$= \left(\frac{2x}{y^2+z^2+2}\,\vec{\imath} + \frac{(y^2+z^2+2)\cdot 2y - (x^2+y^2+1)\cdot 2y}{(y^2+z^2+2)^2}\,\vec{\jmath} + \frac{0-(x^2+y^2+1)\cdot 2z}{(y^2+z^2+2)^2}\,\vec{k} \right)_P =$$

$$= \left(\frac{2x}{y^2+z^2+2}\,\vec{\imath} + \frac{2y(z^2-x^2+1)}{(y^2+z^2+2)^2}\,\vec{\jmath} - \frac{2z(x^2+y^2+1)}{(y^2+z^2+2)^2}\,\vec{k} \right)_P =$$

$$= \frac{6}{4+1+2}\,\vec{\imath} + \frac{-4(1-9+1)}{(4+1+2)^2}\,\vec{\jmath} - \frac{2(9+4+1)}{(4+1+2)^2}\,\vec{k} = \underline{\underline{\frac{6}{7}\,\vec{\imath} + \frac{4}{7}\,\vec{\jmath} - \frac{4}{7}\,\vec{k}}}\,.$$

$f(P) = \dfrac{3^2+(-2)^2+1}{(-2)^2+1^2+2} = 2$, så en likning for nivåflaten er: $\underline{\underline{\dfrac{x^2+y^2+1}{y^2+z^2+2} = 2}}$,

$$\text{dvs.:}\quad \underline{\underline{x^2-y^2-2z^2=3}}\,.$$

En likning for tangentplanet er: $\underline{\underline{\dfrac{6}{7}(x-3)+\dfrac{4}{7}(y+2)-\dfrac{4}{7}(z-1)=2}}$,

$$\text{dvs.:}\quad \underline{\underline{3x+2y-2z=3}}\,.$$

(b) $$\vec{u} = \frac{\nabla f(P)}{||\nabla f(P)||} = \frac{\frac{6}{7}\,\vec{\imath} + \frac{4}{7}\,\vec{\jmath} - \frac{4}{7}\,\vec{k}}{\sqrt{\frac{36}{49}+\frac{16}{49}+\frac{16}{49}}} = \frac{\frac{6}{7}\,\vec{\imath} + \frac{4}{7}\,\vec{\jmath} - \frac{4}{7}\,\vec{k}}{\sqrt{68}/7} = \frac{6}{\sqrt{68}}\,\vec{\imath} + \frac{4}{\sqrt{68}}\,\vec{\jmath} - \frac{4}{\sqrt{68}}\,\vec{k} =$$

$$= \underline{\underline{\frac{3}{\sqrt{17}}\,\vec{\imath} + \frac{2}{\sqrt{17}}\,\vec{\jmath} - \frac{2}{\sqrt{17}}\,\vec{k}}}$$

Oppgave 2

(a) $$\iiint_T z^2 dV = \underline{\underline{\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}} z^2 dz\,dy\,dx}}$$

(b) Kuleflaten i sylinderkoordinater: $r^2+(z-2)^2=4$, som gir: $z=2\pm\sqrt{4-r^2}$, og kulas xy-projeksjon er avgrenset av sirkelen $r^2=4$, dvs. $r=2$, og $0\le\theta\le 2\pi$.

$$\iiint_T z^2 dV = \underline{\underline{\int_0^{2\pi}\int_0^{2}\int_{2-\sqrt{4-r^2}}^{2+\sqrt{4-r^2}} z^2 r\,dz\,dr\,d\theta}}$$

(c) Kuleflaten i kulekoordinater: $\rho^2 = 4z = 4\rho\cos\phi$, som gir: $\rho = 4\cos\phi$.

$$\iiint_T z^2 dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^{4\cos\phi} (\rho\cos\phi)^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta =$$

$$= \underline{\underline{\int_0^{2\pi} \int_0^{\pi/2} \int_0^{4\cos\phi} \rho^4 \cos^2\phi \sin\phi \, d\rho \, d\phi \, d\theta}}$$

(d) $\iiint_T z^2 dV = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{4\cos\phi} (\rho\cos\phi)^2 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta =$

$$= \int_0^{2\pi} \int_0^{\pi/2} \cos^2\phi \sin\phi \int_0^{4\cos\phi} \rho^4 \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/2} \cos^2\phi \sin\phi \cdot \frac{1}{5}(4\cos\phi)^5 \, d\phi \, d\theta =$$

$$= \frac{1024}{5} \int_0^{2\pi} \int_0^{\pi/2} \cos^7\phi \sin\phi \, d\phi \, d\theta = \frac{1024}{5} \int_0^{2\pi} \left[-\frac{1}{8} \cos^8\phi \right]_0^{\pi/2} d\theta =$$

$$= -\frac{128}{5} \int_0^{2\pi} \left[\cos^8\phi \right]_0^{\pi/2} d\theta = -\frac{128}{5} \int_0^{2\pi} (0-1) \, d\theta = \frac{128}{5} \cdot 2\pi = \underline{\underline{\frac{256\pi}{5}}}$$

Oppgave 3

(a) $C_1 : \vec{r}(t) = 3t\vec{\imath} + 5t\vec{\jmath} + t\vec{k}$, $t : 0 \to 1$, gir: $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) \, dt =$

$$= \int_0^1 \left(\frac{9}{2} \cdot 25t^2 \, \vec{\imath} + (1 - 9t^2 - 25t^2) \, \vec{\jmath} + (1 - 9t^2 - 25t^2) \, \vec{k} \right) \cdot (3\vec{\imath} + 5\vec{\jmath} + \vec{k}) \, dt =$$

$$= \int_0^1 \left(\frac{675}{2} t^2 + 5(1 - 34t^2) + (1 - 34t^2) \right) dt = \int_0^1 (6 + \frac{267}{2} t^2) dt =$$

$$= 6 + \frac{267}{6} = \underline{\underline{\frac{101}{2}}}$$

(b) S er grafen til $z = 1 - \frac{1}{3}x - \frac{1}{2}y$, med $0 \leq y \leq (2 - \frac{2}{3}x)$ og $0 \leq x \leq 3$, slik at

$$\iint_S \vec{F} \cdot \vec{n} \, dS =$$

$$= \int_0^3 \int_0^{2 - \frac{2}{3}x} \left(\frac{9}{2} y^2 \, \vec{\imath} + (1 - x^2 - y^2) \, \vec{\jmath} + (1 - x^2 - y^2) \, \vec{k} \right) \cdot \left(\frac{1}{3} \vec{\imath} + \frac{1}{2} \vec{\jmath} + \vec{k} \right) dy \, dx =$$

$$= \int_0^3 \int_0^{2 - \frac{2}{3}x} \left(\frac{3}{2} y^2 + \frac{1}{2}(1 - x^2 - y^2) + (1 - x^2 - y^2) \right) dy \, dx =$$

$$= \frac{3}{2} \int_0^3 \int_0^{2 - \frac{2}{3}x} (1 - x^2) \, dy \, dx = \frac{3}{2} \int_0^3 \left[(1 - x^2) y \right]_{y=0}^{y = 2 - \frac{2}{3}x} dx =$$

$$= \frac{3}{2} \int_0^3 (1 - x^2)(2 - \frac{2}{3}x) \, dx = \int_0^3 (1 - x^2)(3 - x) \, dx =$$

$$= \int_0^3 (3 - x - 3x^2 + x^3) \, dx = \left[3x - \frac{1}{2}x^2 - x^3 + \frac{1}{4}x^4 \right]_0^3 =$$

$$= 9 - \frac{9}{2} - 27 + \frac{81}{4}) = \frac{63}{4} - 18 = \underline{\underline{-\frac{9}{4}}}$$

(c) $\nabla \cdot \vec{F} = 0 - 2y + 0 = \underline{\underline{-2y}}$

Beregning av trippelintergralet <u>uten</u> bruk av Divergenssetningen, med bruk av sylinderkoordinater:

$$\iiint_T \nabla \cdot \vec{F}\, dV = \iiint_T (-2y)\, dV = -2 \int_0^{2\pi} \int_0^1 \int_0^1 r \sin\theta \; r\, dz\, dr\, d\theta =$$

$$= -2 \int_0^1 dz \cdot \int_0^1 r^2 dr \cdot \int_0^{2\pi} \sin\theta\, d\theta = -2 \cdot 1 \cdot \frac{1}{3} \cdot 0 = \underline{\underline{0}}$$

<u>Med</u> Divergenssetningen, og med S_T lik hele overflaten til T, orientert ut fra T:

$$\iiint_T \nabla \cdot \vec{F}\, dV = \iint_{S_T} \vec{F} \cdot \vec{n}\, dS =$$

$$= \iint_{S_{\text{bunn}}} \vec{F} \cdot \vec{n}\, dS + \iint_{S_{\text{topp}}} \vec{F} \cdot \vec{n}\, dS + \iint_{S_{\text{syl}}} \vec{F} \cdot \vec{n}\, dS =$$

$$= \iint_{T_{xy}} \vec{F}_{z=0} \cdot (-\vec{k})\, dA + \iint_{T_{xy}} \vec{F}_{z=1} \cdot \vec{k}\, dA + \iint_{S_{\text{syl}}} \vec{F} \cdot \vec{n}\, dS =$$

$$= -\iint_{T_{xy}} (1 - x^2 - y^2) dA + \iint_{T_{xy}} (1 - x^2 - y^2) dA + \iint_{S_{\text{syl}}} \vec{F} \cdot \vec{n}\, dS =$$

$$= \iint_{S_{\text{syl}}} \vec{F} \cdot \vec{n}\, dS = \iint_{S_{\text{syl}}} \frac{9}{2} y^2\, \vec{\imath} \cdot (x\vec{\imath} + y\vec{\jmath})\, dS = \frac{9}{2} \iint_{S_{\text{syl}}} y^2 x\, dS =$$

$$= \frac{9}{2} \int_0^1 \int_0^{2\pi} \sin^2\theta \cos\theta\, d\theta\, dz = \frac{9}{2} \int_0^1 dz \cdot \left[\frac{1}{3} \sin^3\theta\right]_0^{2\pi} = \underline{\underline{0}}\ ,$$

fordi på sylinderflaten S_{syl} er $x^2 + y^2 = 1$, slik at $\vec{F}(x, y, z)$ her er lik $\frac{9}{2} y^2\, \vec{\imath}$, mens $\vec{n} = x\vec{\imath} + y\vec{\jmath}$, og S_{syl} kan parametriseres med: $\vec{r}(\theta, z) = \cos\theta\, \vec{\imath} + \sin\theta\, \vec{\jmath} + z\vec{k}$ med $0 \leq \theta \leq 2\pi$ og $0 \leq z \leq 1$.

(d) Vi bruker Stokes' teorem, med S_2 lik den begrensede delen av sylinderflaten $x^2 + y^2 = 1$ som har C_2 som randkurve. Den angitte orienteringen av C_2 svarer da til at S_2 er orientert med $\vec{n}$ rettet ut fra sylinderen $x^2 + y^2 \leq 1$. Videre er

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{9}{2} y^2 & 1 - x^2 - y^2 & 1 - x^2 - y^2 \end{vmatrix} = -2y\, \vec{\imath} + 2x\, \vec{\jmath} + (-2x + 9y)\, \vec{k}\ ,$$

og $\vec{n} = x\vec{\imath} + y\vec{\jmath}$ (liksom for flaten S_{syl} i (c) over), slik at

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \iint_{S_2} \nabla \times \vec{F} \cdot \vec{n}\, dS = \iint_{S_2} \left(-2y\, \vec{\imath} + 2x\, \vec{\jmath} + (-2x + 9y)\, \vec{k}\right) \cdot (x\vec{\imath} + y\vec{\jmath})\, dS =$$

$$= \iint_{S_2} (-2yx + 2xy)\, dS = \iint_{S_2} 0\, dS = \underline{\underline{0}}\ ,$$

Oppgave 4

Strengens midtpunkt holder seg helt i ro. Dette kan vi se f.eks. av d'Alemberts løsning, idet den halvperiodiske utvidelsen $\tilde{f}$ av f er: $\tilde{f}(x) = \frac{1}{1000}\sin 2\pi x$ for $x \in \mathbf{R}$:

$$u(x,t) = \frac{1}{2}(\tilde{f}(x-ct) + \tilde{f}(x+ct)) = \frac{1}{2000}(\sin 2\pi(x-ct) + \sin 2\pi(x+ct)) ,$$

slik at

$$u(\frac{1}{2},t) = \frac{1}{2000}(\sin 2\pi(\frac{1}{2}-ct) + \sin 2\pi(\frac{1}{2}+ct)) = \frac{1}{2000}(\sin(\pi - 2\pi ct) + \sin(\pi + 2\pi ct)) =$$

$$= \frac{1}{2000}(\sin\pi\cos 2\pi ct - \cos\pi\sin 2\pi ct + \sin\pi\cos 2\pi ct + \cos\pi\sin 2\pi ct) = 0 .$$