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Oppgave 1

(a) $f_x(x,y) = \dfrac{(x^2+4)\cdot 2x - (x^2+2y^2+3)\cdot 2x}{(x^2+4)^2} = \dfrac{2x(1-2y^2)}{(x^2+4)^2}$

og $f_y(x,y) = \dfrac{(x^2+4)\cdot 4y - (x^2+2y^2+3)\cdot 0}{(x^2+4)^2} = \dfrac{4y}{x^2+4}$, som gir at

$\nabla f(x,y) = f_x(x,y)\,\vec{\imath} + f_y(x,y)\,\vec{\jmath} = \underline{\underline{\dfrac{2x(1-2y^2)}{(x^2+4)^2}\,\vec{\imath} + \dfrac{4y}{x^2+4}\,\vec{\jmath}}}$.

Den enhetsvektoren som peker i retningen for maksimal $f(x,y)$-stigning i P er:

$\vec{u} = \dfrac{\nabla f(2,1)}{|\nabla f(2,1)|} = \dfrac{-\frac{1}{16}\vec{\imath} + \frac{1}{2}\vec{\jmath}}{|-\frac{1}{16}\vec{\imath} + \frac{1}{2}\vec{\jmath}|} = \dfrac{-\vec{\imath}+8\vec{\jmath}}{|-\vec{\imath}+8\vec{\jmath}|} = \dfrac{-\vec{\imath}+8\vec{\jmath}}{\sqrt{65}} = \underline{\underline{-\dfrac{1}{\sqrt{65}}\vec{\imath} + \dfrac{8}{\sqrt{65}}\vec{\jmath}}}$.

(b) $-\vec{\imath}+8\vec{\jmath}$ er en normalvektor til tangenten, så en likning kan settes opp slik:

$$\underline{\underline{-(x-2)+8(y-1)=0}} \qquad \text{eller:} \qquad \underline{\underline{y = \frac{1}{8}x + \frac{3}{4}}}\ .$$

(c) Linearisering betyr "følg tangenten i P", som her betyr at vi får y-verdien

$$y \approx \frac{1}{8}\cdot 2{,}001 + \frac{3}{4} \approx \underline{\underline{1{,}0001}}\ .$$

Oppgave 2

(a) $V_T = \iint_D xy\,dA = \int_0^2 \int_0^{\sqrt{4-x^2}} xy\,dy\,dx = \int_0^2 \left[\frac{1}{2}xy^2\right]_0^{\sqrt{4-x^2}} dx =$

$$= \int_0^2 \frac{1}{2}x(4-x^2)dx = \int_0^2 (2x - \frac{1}{2}x^3)dx = \left[x^2 - \frac{1}{8}x^4\right]_0^2 = \underline{\underline{2}}\ .$$

(b) $A_S = \iint_D \sqrt{1+(f_x(x,y))^2+(f_y(x,y))^2}\,dA = \iint_D \sqrt{1+y^2+x^2}\,dA =$

$$= \int_0^{\pi/2}\int_0^2 \sqrt{1+r^2}\,r\,dr\,d\theta = \frac{\pi}{2}\int_1^5 \sqrt{u}\cdot\frac{1}{2}\,du = \frac{\pi}{4}\left[\frac{2}{3}u^{3/2}\right]_1^5 = \underline{\underline{\frac{\pi}{6}(5\sqrt{5}-1)}}\ .$$

(c) Den vertikale sideveggen er en del av flaten $r=2$, så den kan parametriseres med θ og z, slik:

$$\vec{r}(\theta,z) = 2\cos\theta\,\vec{\imath} + 2\sin\theta\,\vec{\jmath} + z\,\vec{k} \quad \text{med} \quad 0 \le z \le 4\cos\theta\sin\theta \quad \text{og} \quad 0 \le \theta \le \tfrac{\pi}{2}\ .$$

Dette gir: $\dfrac{\partial\vec{r}}{\partial\theta} \times \dfrac{\partial\vec{r}}{\partial z} = (-2\sin\theta\,\vec{\imath} + 2\cos\theta\,\vec{\jmath}) \times \vec{k} = 2\cos\theta\,\vec{\imath} + 2\sin\theta\,\vec{\jmath}$, slik at

$\left|\dfrac{\partial\vec{r}}{\partial\theta} \times \dfrac{\partial\vec{r}}{\partial z}\right| = \sqrt{4\cos^2\theta + 4\sin^2\theta} = 2$, og dermed er

$$A_{\text{sidevegg}} = \int_0^{\pi/2}\int_0^{4\cos\theta\sin\theta} 2\,dz\,d\theta = 8\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta =$$

$$= 4\int_0^{\pi/2} \sin 2\theta \, d\theta = 4\left[-\frac{1}{2}\cos 2\theta\right]_0^{\pi/2} = -2(-1-1) = \underline{\underline{4}}\,.$$

Oppgave 3

$$V_T = \iiint_T dV = \iiint_T \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi}\int_0^{2+\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta =$$

$$= 2\pi \cdot \int_0^{\pi} \sin\phi \int_0^{2+\cos\phi} \rho^2 \, d\rho \, d\phi = 2\pi \cdot \int_0^{\pi} \sin\phi \cdot \frac{1}{3}(2+\cos\phi)^3 d\phi =$$

$$= -\frac{2}{3}\pi \int_3^1 u^3 du = -\frac{2}{3}\pi \left[\frac{1}{4}u^4\right]_3^1 = -\frac{\pi}{6}(1^4 - 3^4) = \frac{80}{6}\pi = \underline{\underline{\frac{40}{3}\pi}}\,.$$

Oppgave 4

(a) $\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_T \nabla\cdot\vec{F}\, dV = \iiint_T 0\, dV = \underline{\underline{0}}.$

(b) $\iint_{S_{\text{bunn}}} \vec{F}\cdot\vec{n}\, dS = \iint_{S_{\text{topp}}} \vec{F}\cdot(-\vec{k})\, dS = -\iint_{S_{\text{topp}}} x\, dS = -\int_0^{2\pi}\int_0^{\sqrt{3}} r\cos\theta\cdot r\, dr\, d\theta = \underline{\underline{0}},$

$$\iint_{S_{\text{topp}}} \vec{F}\cdot\vec{n}\, dS = \iint_{S_{\text{topp}}} \vec{F}\cdot\vec{k}\, dS = \iint_{S_{\text{topp}}} x\, dS = \int_0^{2\pi}\int_0^{\sqrt{3}} r\cos\theta\cdot r\, dr\, d\theta = \underline{\underline{0}}$$

og $\iint_{S_{\text{syl}}} \vec{F}\cdot\vec{n}\, dS = \iint_S \vec{F}\cdot\vec{n}\, dS - (\iint_{S_{\text{bunn}}} \vec{F}\cdot\vec{n}\, dS + \iint_{S_{\text{topp}}} \vec{F}\cdot\vec{n}\, dS) = 0 - (0+0) = \underline{\underline{0}}\,.$

(c) $\iint_{S_{\text{topp}}} \nabla\times\vec{F}\cdot\vec{n}\, dS = \iint_{S_{\text{topp}}} (-\vec{\imath}-\vec{\jmath}-\vec{k})\cdot\vec{k}\, dS = -\iint_{S_{\text{topp}}} dS = -A_{S_{\text{topp}}} = \underline{\underline{-3\pi}}.$

(d) Den angitte retningen langs C er den positive orienteringen av C m.h.t. orienteringen av S_{topp} gitt ved $\vec{n}$, så Stokes' teorem gir oss at $\int_C \vec{F}\cdot\vec{T}\, ds = \iint_{S_{\text{topp}}} \nabla\times\vec{F}\cdot\vec{n}\, dS = \underline{\underline{-3\pi}}.$

Oppgave 5

Separasjon av variable gir oss disse løsningene på formen $u(x,t) = X(x)T(t)$ som har $u(0,t) = u(1,t) = 0$ og $u_t(x,0) = 0$ for alle $t \geq 0$: $u(x,t) = C\sin n\pi x \cos n\pi t$ med $n = 1,2,3,4,\ldots$. Uendelig mange av disse har $u(\frac{1}{3},t) = u(\frac{1}{2},t) = 0$ for alle $t \geq 0$ — den første er $u(x,t) = C\sin 6\pi x \cos 6\pi t$. Svaret er altså: $\underline{\underline{\text{Ja}}}$, bølgelikningen $u_{xx} = u_{tt}$ har slike løsninger.