TECHNICAL COMMENT

Some Remarks on Bad Convex Functions

Franco Giannessi*

Dedicated to Professor Christiane Tammer on the occasion of her 65th birthday

Abstract. This note deals shortly with some questions about a class of convex functions, which have a special behaviour and which have been named bad.

1. A Question

Let $f : \mathbb{R}^n \to \mathbb{R}$, with $n \geq 2$, be a convex function, and set:

$$x(t) = (t,\ 0,\ .....,0), \quad \textit{with}\ \ t \in \mathbb{R} . \tag{1}$$

Assume that, $\forall t > 0$, the gradient $\nabla f(x(t))$ exists. The question consists in asking whether there exist functions f, such that the following limit

$$\lim_{t \downarrow 0} \nabla f(x(t)) \tag{2}$$

does not exist. The question appeared in [1]. Ennio De Giorgi was the first to answer in the affirmative, constructing a function f, without demonstration (see [2] for historical details); it, together with another function f, was given a few later by Tyrrell Rockafellar; both facts have been published in [3]. Independently of them and of each other, other mathematicians have discovered further functions f, naming them *bad convex functions*; see, e.g., [4].
Several questions are still open. When dealing with numerical methods, perhaps, to have to deal with a bad convex function might be umpleasant. Therefore, a condition for a convex function to be bad would be desirable.

----

* Department of Mathematics, University of Pisa, Pisa, Italy

What happens if $x(t)$ is a curve, having the origin as endpoint, but being not necessarily a ray? Do such functions, with $x(t)$ a ray or not, exist, if we pose ourselves in infinite dimensional spaces?

2. A Further Question

Let $f : \mathbb{R}^n \to \mathbb{R}$, with $n \geq 2$, be as above and consider again (1). Let H denote the Hessian. Now assume that, $\forall t > 0$, both $\nabla f(x(t))$ and $Hf(x(t))$ exist. May a function f exist, such that both limits (2) and

$$\lim_{t \downarrow 0} Hf(x(t)) \tag{3}$$

do not exist?
The question extends naturally to higher orders. Restricting ourselves to analytic functions, the case of infinite orders might be treatable.
All the questions of the previous section can be considered here too.
Recently [5], Dariusz Zagrodny has made very interesting insights and observed that answers to the above questions might help with other pending questions, known in the literature; e.g., Klee's problems: the convexity of Chebyshev sets and farthest point conjecture; Borwein's question on the Alexandrov Theorem in infinite dimensions.

References

[1] F.Giannessi, A Problem in Convex Functions, JOTA, Vol.59, No.5, 1988, p.525.
[2] M. Théra, "Every mathematician can do a true theorem. Only a genius can make an important mistake" (interview). JOTA Special Issue on "Optimization, Variational Analysis and Applications". To appear.
[3] R.T. Rockafellar, On a Special Class of Convex Functions, JOTA, Vol.70, No.3, 1991, pp.619-621.
[4] D.Zagrodny, An example of Bad Convex Function, JOTA, Vo.70, No.3, 1991, pp.631-637.

[5] D. Zagrodny, “The strong convergence of subgradients of convex functions along directions, perspectives and open problems,” JOTA, Vol. 178, No. 2, 2018.