Some examples of sms-algebras

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Abelian categories and triangulated categories

From abelian categories to triangulated categories:

$$\mathcal{A} \rightsquigarrow \mathrm{K}^*(\mathcal{A}),\ \mathcal{D}^*(\mathcal{A}),\ \mathcal{D}_{\mathrm{sg}}(\mathcal{A}),\ \mathcal{C}_{\mathcal{A}}, \ldots$$

From triangulated categories to abelian categories:

$$\mathcal{T} \rightsquigarrow\ ???$$

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Two known methods

k: field
$\mathcal{T}$: triangulated k-category with shift functor Σ

Beilinson-Bernstein-Deligne

If $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ is a t-structure, then the heart $\mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$ is an abelian subcategory.

Buan-Marsh-Reiten, Keller-Reiten, Koenig-Zhu, Iyama-Yoshino

Let $\mathcal{M}$ be a contravariantly finite rigid subcategory of $\mathcal{T}$. Then there is an equivalence

$$\frac{\mathcal{M} * \Sigma\mathcal{M}}{[\Sigma\mathcal{M}]} \xrightarrow{\simeq} \operatorname{mod} \mathcal{M}.$$

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A new method I

$w \geq 2$: integer
Assume: $\mathcal{T}$ is Hom-finite Krull–Schmidt

Definition (Coelho Simões)

A collection $\mathcal{S}$ of objects of $\mathcal{T}$ is called a w-simple-minded system if

(1) $\mathrm{Hom}(S, S') = \begin{cases} \text{a skew field} & \text{if } S = S' \\ 0 & \text{otherwise} \end{cases},$

(2) $\mathrm{Hom}(S, \Sigma^p S') = 0 \quad \forall\, S, S' \in \mathcal{S}$ and $-w+1 \leq p \leq -1$,

(3) $\mathcal{T} = \langle \mathcal{S} \rangle * \Sigma^{-1}\langle \mathcal{S} \rangle * \cdots * \Sigma^{-w+1}\langle \mathcal{S} \rangle$.

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A new method II

$\mathcal{S}$: w-simple-minded system

Theorem (Jørgensen for $w \geq 2$, Iyama-Jin for $w \geq 3$)

$\langle \mathcal{S} \rangle$ is an abelian category and there is an equivalence

$$\langle \mathcal{S} \rangle \simeq \operatorname{mod} A$$

for some finite-dimensional k-algebra A.

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Negative cluster categories

Q: acyclic quiver

Definition (Keller)

The $(-w)$-cluster category is

$$\mathcal{C}_{-w}(Q) := \mathcal{D}^b(kQ)/\tau\Sigma^{w+1}.$$

Facts

- $\mathcal{C}_{-w}(Q)$ is a $(-w)$-Calabi–Yau triangulated category.
- The simple kQ-modules $\{S_1, \ldots, S_n\}$ is a w-simple-minded system in $\mathcal{C}_{-w}(Q)$, and $\langle S_1, \ldots, S_n \rangle = \operatorname{mod} kQ$.

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sms-algebras I

$w = 2$
A: finite-dimensional k-algebra

Defintion

A is called an sms-algebra of type Q if

$$\operatorname{mod} A \simeq \langle \mathcal{S} \rangle$$

for some 2-simple-minded system $\mathcal{S}$ of $\mathcal{C}_{-2}(Q)$.

Example

kQ is an sms-algebra.

Examples? Properties? Classification?

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Theorem (Xie-Y)

If $T \in \mathcal{D}^b(kQ)$ is a tilting complex with $H^{\neq 0,-1}(T) = 0$, then $\mathrm{End}(T)$ is an sms-algebra.

Theorem (Xie-Y)

The following conditions are equivalent for a finite-dimensional algebra A:

(i) A is an sms-algebra of type $\mathbb{A}$

(ii) A is a connected gentle algebra without cycles.

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