On derived Picard groups of quivers

Flash Talks in Representation Theory

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Picard groups

- If $(\mathcal{C}, \otimes, \mathbb{1})$ is a monoidal category, we call an object X invertible if $X \otimes Y \cong \mathbb{1}$ for some object Y.
- The Picard group is defined as $\mathrm{Pic}(\mathcal{T}) = \{X \in \mathcal{T} \text{ invertible}\}/\cong$.
- Classical situation: $(\mathrm{coh}\,\mathbb{X}, \otimes, \mathcal{O}_{\mathbb{X}})$. Invertible sheaves correspond to line bundles.

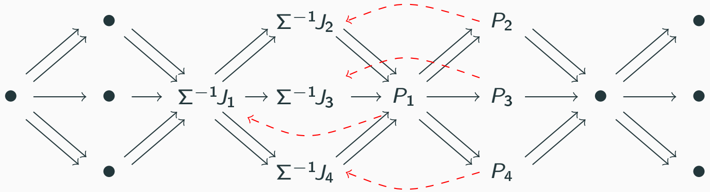
- Representation theory: $\big(\mathrm{mod}(KQ \otimes KQ^{\mathrm{op}}), \otimes_{KQ}, KQ\big)$, where K a commutative ring, Q a finite acyclic quiver.
- Derived version ([Miyachi-Yekutieli 2001] for K a field): $\big(\mathrm{D}^{\mathrm{b}}(KQ \otimes KQ^{\mathrm{op}}), \otimes^{\mathbb{L}}_{KQ}, KQ\big)$.
- Remark: There is a group homomorphism (iso for K a field)

$$\mathrm{Pic}\big(\mathrm{D}^{\mathrm{b}}(KQ \otimes KQ^{\mathrm{op}})\big) \longrightarrow \mathrm{AutEq}_{\Delta}\big(\mathrm{D}^{\mathrm{b}}(KQ)\big)/\cong$$

$$T \longmapsto T \otimes^{\mathbb{L}}_{KQ} -$$

Reminder on the derived category for a field K

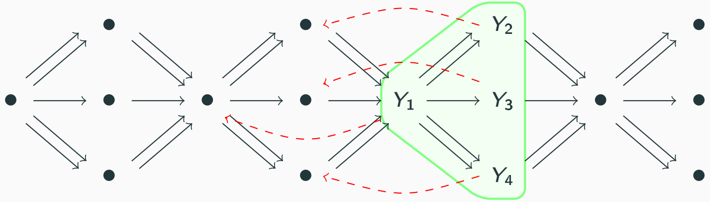
- Consider a transjective component of the Auslander-Reiten quiver of $\mathrm{D}^{\mathrm{b}}(KQ)$, for example:



- This is a so-called repetitive quiver $\mathbb{Z}Q$ of Q, and has a structure of a translation quiver with polarization:
  - there is an automorphism τ of the quiver given by the Auslander-Reiten translation (dotted red arrows)
  - and there is a specified bijection between arrows of the form $Y \to X$ and those of the form $\tau X \to Y$ (polarization).
- We also have Auslander-Reiten triangles in $\mathrm{D}^{\mathrm{b}}(KQ)$ of the form $\tau X \to \bigoplus E_i \to X \xrightarrow{+}$.

Mesh representations of repetitive quivers

- Consider now any triangulated category $\mathcal{T}$, a finite acyclic quiver Q, and the full subcategory of $\mathcal{T}^{\mathbb{Z}Q}$

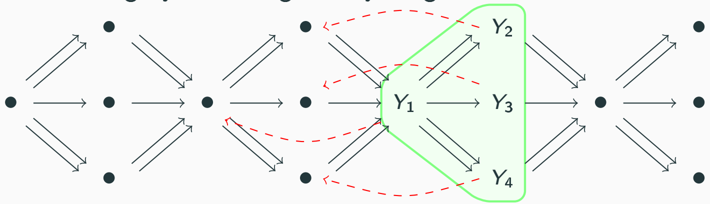


given by diagrams where meshes assemble to triangles of the form $\tau X \to \bigoplus E_i \to X \xrightarrow{+}$.

- Denote this subcategory by $\mathcal{T}^{\mathbb{Z}Q,\mathrm{mesh}}$.
- Observation: Every object of $\mathcal{T}^{\mathbb{Z}Q,\mathrm{mesh}}$ is determined up to isomorphism by its restriction to $\mathcal{T}^{Q}$ (the green area).
- In fact, the restriction functor $\mathrm{res}\colon \mathcal{T}^{\mathbb{Z}Q,\mathrm{mesh}} \to \mathcal{T}^{Q}$ is surjective on objects, but has no reason to be an equivalence.

Stable ∞-categories and the equivalence

- Replace $\mathcal{T}$ be a stable ∞-category $\mathcal{C}$ in the sense of Lurie (its homotopy category $h\mathcal{C}$ is then triangulated).
- Again, let Q be a finite acyclic quiver Q and $\mathcal{C}^{\mathbb{Z}Q,\mathrm{mesh}}$ be full ∞-subcategory of $\mathcal{C}^{\mathbb{Z}Q}$ given by diagrams

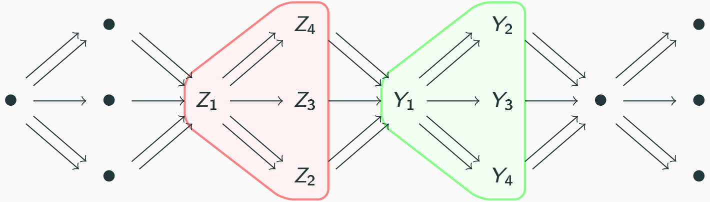


give by diagrams where meshes induce cofiber sequences $\tau X \to \bigoplus E_i \to X$.

Theorem (Á. Sánchez in progress, [Rahn-Š. 2016] for Dynkin type A_n)
The restriction $\mathcal{C}^{\mathbb{Z}Q,\mathrm{mesh}} \to \mathcal{C}^{Q}$ is an equivalence of ∞-categories.

Picard groups and repetitive quivers

- If $\mathcal{C}$ is a stable ∞-category, then $\mathcal{C}^{\mathbb{Z}Q,\mathrm{mesh}} \simeq \mathcal{C}^{Q}$.
- Let $G = \mathrm{Aut}(\mathbb{Z}Q)$ (the group of automorphisms of the repetitive quiver as a translation quiver with polarization).
- Then G acts on $\mathcal{C}^{\mathbb{Z}Q,\mathrm{mesh}}$ canonically.
- Via the equivalence, we can transfer the action to $G \circlearrowright \mathcal{C}^{Q}$:



- Back to the classical case:
  - G also acts on the (triangulated) homotopy category $h(\mathcal{C}^{Q})$.
  - If K is a field and we put $\mathcal{C} = \mathrm{D}^{\mathrm{b}}_{\infty}(K)$, then

$$h(\mathcal{C}^{Q}) = h\big(\mathrm{D}^{\mathrm{b}}_{\infty}(K)^{Q}\big) \simeq h\big(\mathrm{D}^{\mathrm{b}}_{\infty}(KQ)\big) = \mathrm{D}^{\mathrm{b}}(KQ).$$

Concluding remarks

- The equivalences of this type can be obtained as a composition by generalized reflection functors ([Rahn-Š. 2016], [Dyckerhoff-Jasso-Walde 2019], ideas going back to [Bernstein-Gelfand-Ponomarev 1973]).
- In fact, $\mathrm{Aut}(\mathbb{Z}Q) \times \mathbb{Z} \circlearrowright \mathcal{C}^Q$, where the second factor acts by powers of Σ.
- One could expect that if Q is Dynkin, the second factor is not necessary. Known if $\mathcal{C} = \mathrm{D}^{\mathrm{b}}_{\infty}(K)$ for a field K (e.g. [Miyachi-Yekutieli 2001]) or if Q is of type A_n ([Rahn-Š. 2016]).
- To resolve this, enough to consider $\mathcal{C} =$ finite spectra. So it is interesting to understand the spectral Picard group of Q.

Thank you for your attention!