Is convex geometry trying to teach us homological algebra?

Flash Talks in Representation Theory

5th January 2024

Report on joint work with Nathan Broomhead, David Ploog and Jon Woolf

Setup

- $\mathcal{D}$ Hom-finite, Krull-Schmidt triangulated category
- $\mathcal{H}$ heart of a bounded t-structure in $\mathcal{D}$
  (can consider $\mathcal{H}$ Hom-finite abelian cat., $\mathcal{D} = D^b(\mathcal{H})$)
- $\Lambda = K(\mathcal{H}) = K(\mathcal{D})$ finite rank (for simplicity)

$$\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$$

$$\Lambda^*_{\mathbb{R}} = \Lambda^* \otimes \mathbb{R} = \mathrm{Hom}(\Lambda, \mathbb{R})$$

__Theorem__ (Broomhead - P - Ploog - Woolf)

1) The set $C(H) = \{ v \in \Lambda^*_{\mathbb{R}} \mid v([h]) \geq 0 \ \forall h \in H \}$ is a closed, strictly convex cone in $\Lambda^*_{\mathbb{R}}$.

2) There is a "dual face fan" $\Sigma(H)$ in $\Lambda^*_{\mathbb{R}}$ whose set of maximal cones is $\{ C(K) \mid K \subseteq H[1] * H \}$.

3) Each cone in $\Sigma(H)$ has the form $C(K/S)$ for some $K \subseteq H[1] * H$ and $S \subseteq K$ a special (= face) Serre subcat.

4) $\Sigma(H)$ does not depend on D.

5) If H is length then $\Sigma(H)$ is complete.

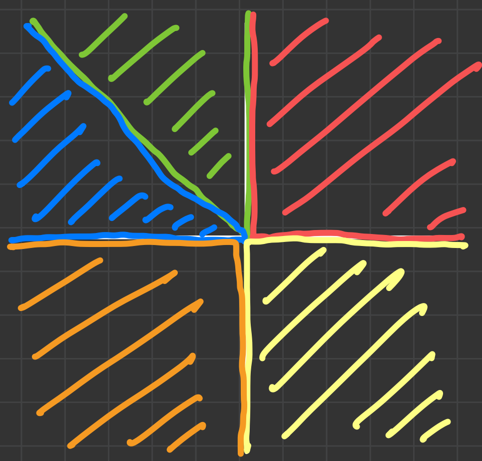
Cones

V f.d. $\mathbb{R}$-vector space.

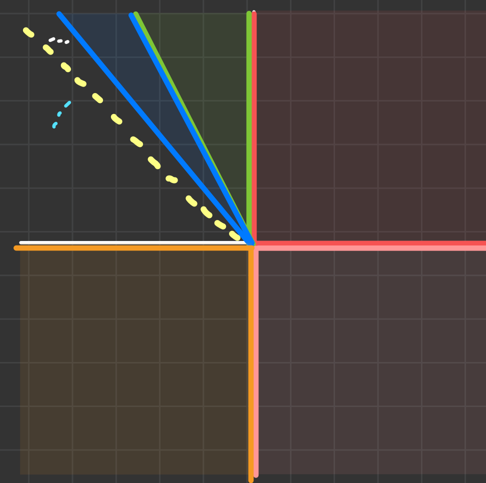
- A cone σ in V is a subset $\sigma + \sigma \subseteq V$
  $\mathbb{R}_{\geq 0} \cdot \sigma \subseteq V$

- A face τ of σ is a subcone s.t. $0 \in \tau$ and $u+v \in \tau \iff u \in \tau$ and $v \in \tau$.

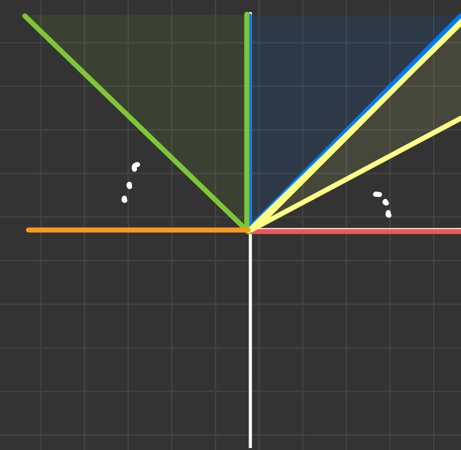
- A fan Σ is a set of cones such that
  a) Σ is closed under taking faces
  b) any two cones intersect in a common face.



g-fan of $1 \to 2$, complete.

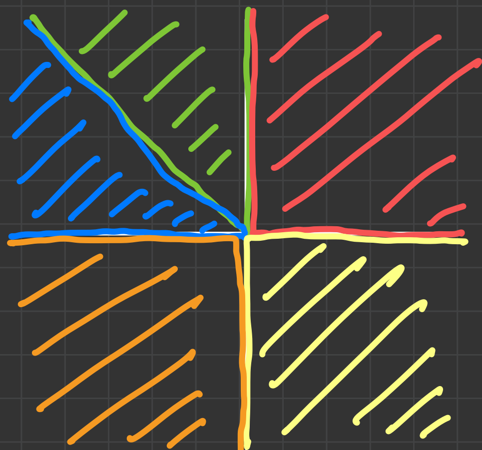


g-fan of $1 \rightrightarrows 2$, not complete.



"g-fan" of coh $\mathbb{P}^1$, definitely not complete!

- A "dual face fan" Σ is a set of cones such that
  a) Σ is closed under taking "dual faces"
  b) any two cones intersect in a common "dual face".



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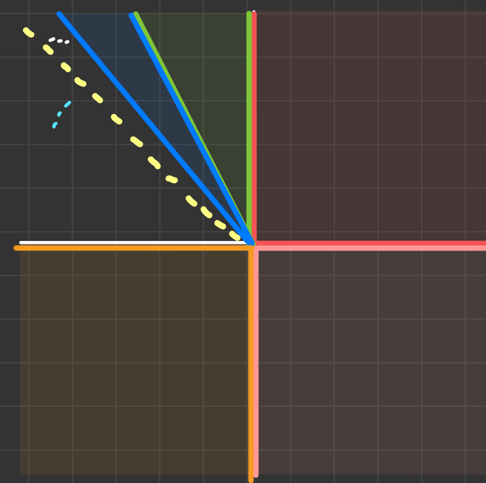

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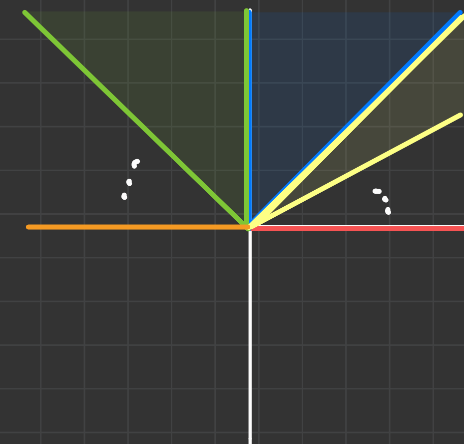


"g-fan" of coh $\mathbb{P}^1$, definitely not complete!

Cones of an abelian category

Effective cone: $E(\mathcal{H}) := \{ a_1[h_1] + \cdots + a_n[h_n] \mid n \in \mathbb{N},\ a_i \geq 0,\ h_i \in \mathcal{H} \} \subseteq \Lambda_{\mathbb{R}}$

Heart cone: $C(\mathcal{H}) := E(\mathcal{H})^{\vee} = \{ v \in \Lambda^*_{\mathbb{R}} \mid v(x) \geq 0\ \forall\, x \in E(\mathcal{H}) \} \subseteq \Lambda^*_{\mathbb{R}}$

Heart dual face fan: $\Sigma(\mathcal{H})$ has maximal cones

$$\{ C(\mathcal{K}) \mid \mathcal{K} \subseteq \mathcal{H}[1] * \mathcal{H} \}.$$

Effective cones for A_2 $\quad 1 \to 2$

$$\begin{tikzcd}[column sep=small]
 & \begin{smallmatrix}1\\2\end{smallmatrix} \arrow[dr] & & 2[1] \arrow[dr] & & 1[1] \\
2 \arrow[ur] & & 1 \arrow[ur] & & \begin{smallmatrix}1\\2\end{smallmatrix}[1] \arrow[ur] &
\end{tikzcd}$$

$H = \langle 1, 2 \rangle$

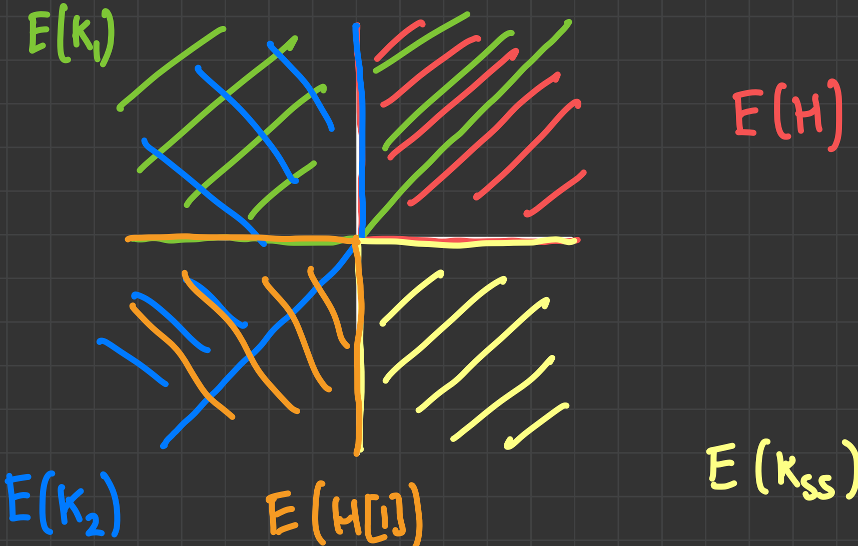
$H[1] = \langle 1[1], 2[1] \rangle$

$K_1 = \langle \begin{smallmatrix}1\\2\end{smallmatrix}, 2[1] \rangle$

$K_2 = \langle 1, \begin{smallmatrix}1\\2\end{smallmatrix}[1] \rangle$

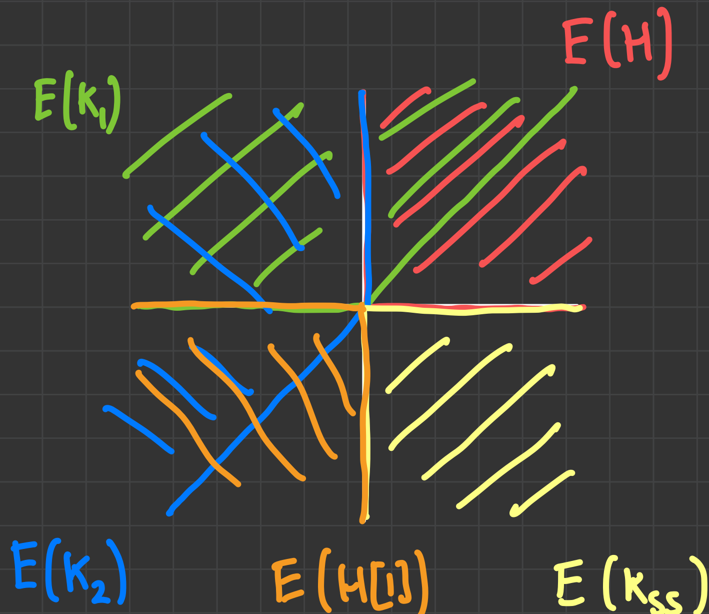
$K_{ss} = \langle 2, 1[1] \rangle$

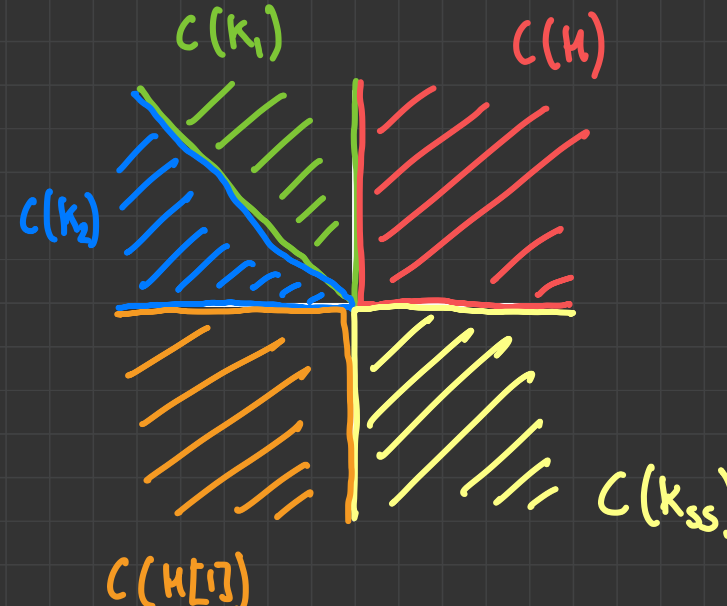
$= add(2, 1[1]).$



NOT A FAN!

The heart fan of A_2





IS A (DUAL FACE) FAN

Recall

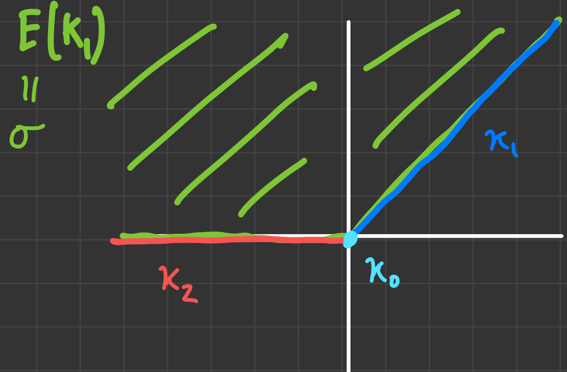
Effective cone: $E(H) := \{ a_1[h_1] + \cdots + a_n[h_n] \mid n \in \mathbb{N},\ a_i \geq 0,\ h_i \in H \} \subseteq \Lambda_{\mathbb{R}}$

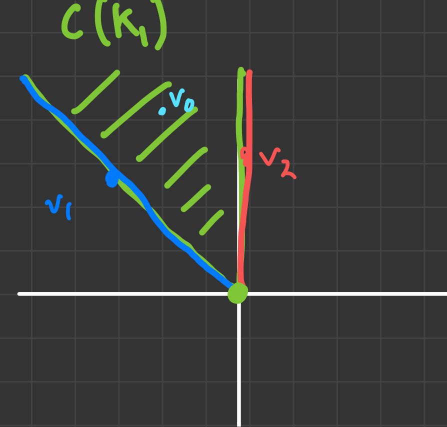
Heart cone: $C(H) := E(H)^\vee = \{ v \in \Lambda_{\mathbb{R}}^* \mid v(x) \geq 0\ \forall x \in E(H) \} \subseteq \Lambda_{\mathbb{R}}^*$

Heart dual face fan: $\Sigma(H)$ has maximal cones

$$\{ C(K) \mid K \subseteq H[1] * H \}.$$

<u>Q^n</u> What are the dual faces/non-maximal cones?





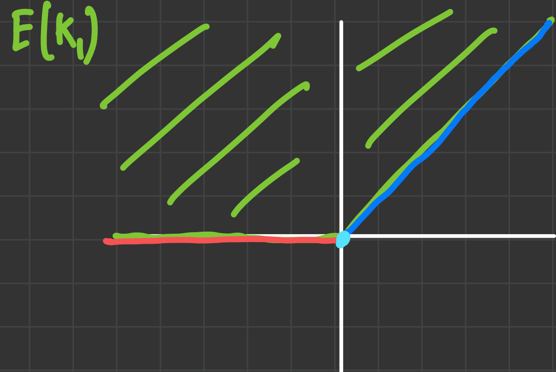
Four "exposed" faces,

$\sigma = E(K_1) \cap 0^{\perp} = E(\ker_{K_1}(0))$ where $\ker_{K_1}(0) = K_1$.

$\kappa_1 = E(K_1) \cap v_1^{\perp} = E(\ker_{K_1}(v_1))$ where $\ker_{K_1}(v_1) = \{ k \in K_1 \mid v_1([k]) = 0 \}$

$\kappa_2 = E(K_1) \cap v_2^{\perp} = E(\ker_{K_2}(v_2))$ where $\ker_{K_1}(v_2) = \{ k \in K_1 \mid v_2([k]) = 0 \}$

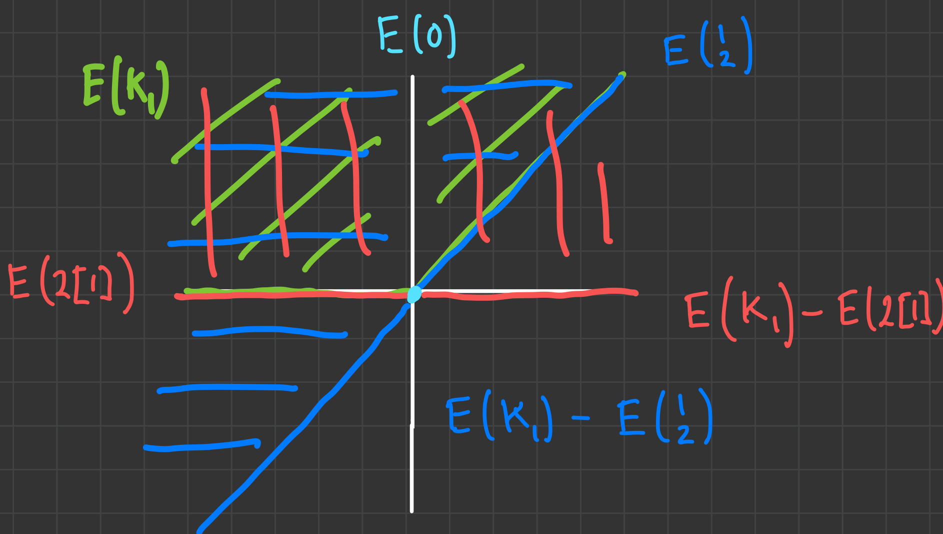
$\kappa_0 = E(K_1) \cap v_0^{\perp} = E([0])$ where $\ker_{K_1}(v_0) = \{ k \in K_1 \mid v_0([k]) = 0 \}$

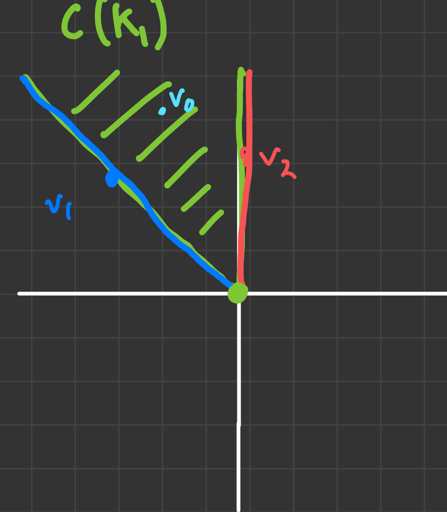


$\ker_{K_1}(0)$, $\ker_{K_1}(v_1)$, $\ker_{K_1}(v_2)$ and $\ker_{K_1}(v_0)$ are Serre subcategories called face subcategories

$$\mathrm{Serre}_\wedge(K) = \{\text{face subcategories of } K\}$$

<u>NB</u> Not all Serre subcategories may occur as face subcats.





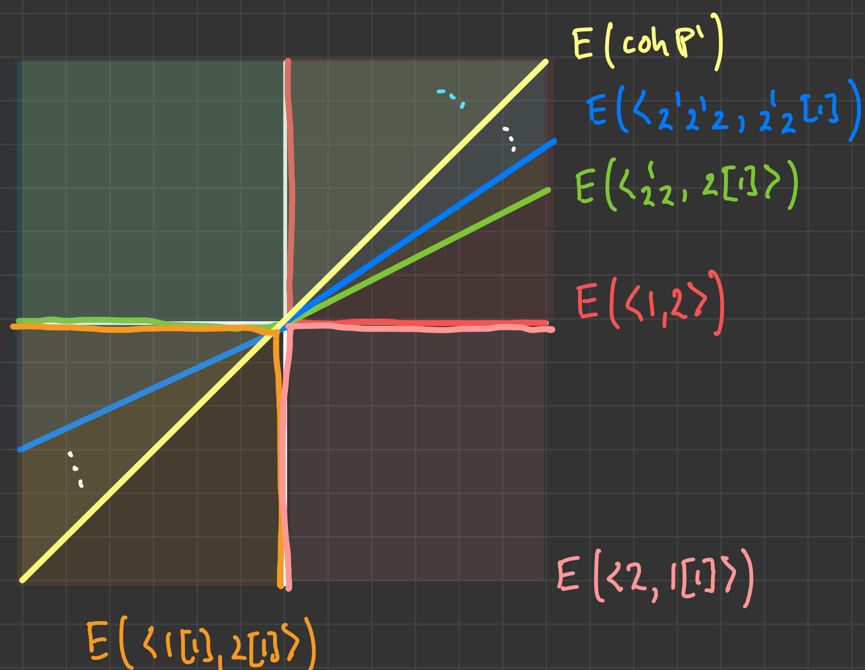
$$\{\text{exposed faces of } E(K_1)\} \xleftrightarrow{1-1} \mathrm{Serre}_\wedge(K)$$

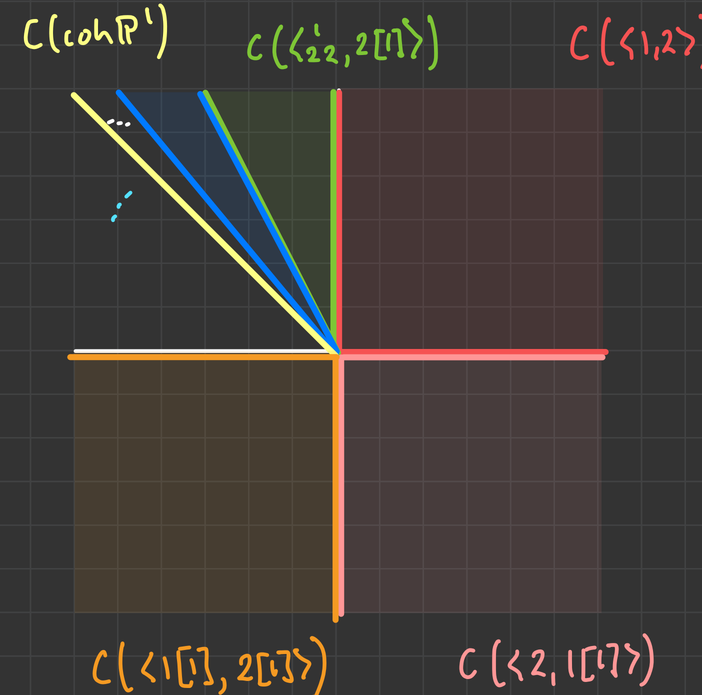
$$E(S) \xleftrightarrow{1-1} S$$

Faces of $C(K)$ are $\boxed{C(K/S) = (E(K) - E(S))^\vee = C(K) \cap E(S)^\perp}$

Example $(1 \rightrightarrows 2)$

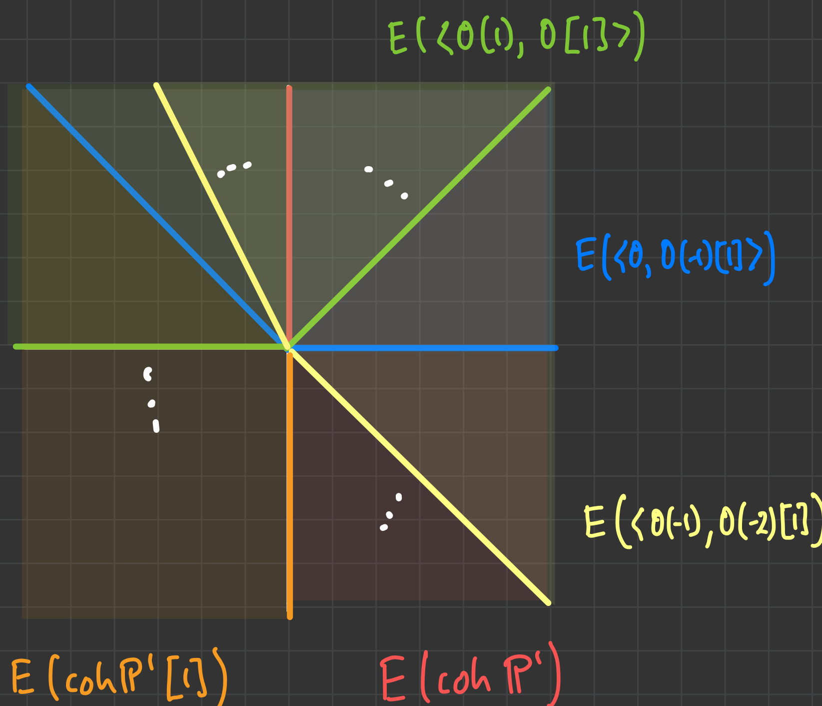
Use basis $S_1 \leftrightarrow (0,1)$ and $S_2 \leftrightarrow (1,0)$

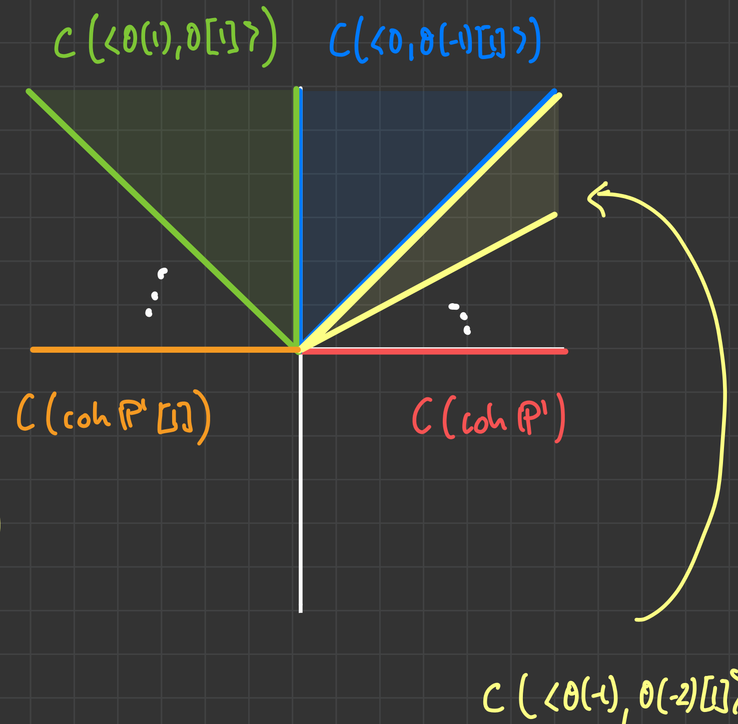




FAN IS COMPLETE!

Example ($\operatorname{coh} \mathbb{P}^1$) Use basis $\mathcal{O} \leftrightarrow (1,0)$ $\mathcal{O}_x \leftrightarrow (0,1)$





NOT COMPLETE

Concluding remarks

- Effective cones are "c-vector cones".
- The heart dual face fan is "the completion of the g-fan".
- Simple objects (and therefore c-vectors) are the more fundamental objects w.r.t. tilting.
- The heart dual face fan "virtual g-fan" – it lets you play with g-vectors even when there aren't any.

Thank you!

Upcoming conference:

"Simple-mindedness, silting and stability"

8–12 July 2024, Ambleside, English Lake District

www.lancaster.ac.uk/maths/research/conferences/stability