On 1-Gorenstein algebras of finite Cohen-Macaulay type

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Motivation

Let Λ be an Artin algebra of finite representation type.

Question. When Auslander algebra of Λ is also of finite representation type?

Auslander (1971) in his Queen Mary College Notes, showed that

'this is the case if and only if $T_2(\Lambda)$ is of finite representation type'

where $T_n(\Lambda)$ denote the algebra of n by n lower triangular matrices.

Auslaner and Reiten (1974): $T_2(\Lambda)$ is of finite representation type, if
- Λ is a radical square zero Nakayama algebra or
- Λ is an algebra such that each indecomposable Λ-module is projective or injective.

Motivation

They also showed that although $T_2(T_2(\Lambda))$ may be of finite representation type, but $T_2(T_2(T_2(\Lambda)))$ is always of infinite representation type.

Incomplete list of references:

Z. Leszczynski and D. Simson (1979), K. Igusa, M. Platzeck, G. Todorov and D. Zacharia (1987), V. Dlab, C.M. Ringel (1990), G. Todorov and D. Zacharia (1991), B-L. Xiong, P. Zhang and Y-H. Zhang (2014), M. Lu (2020), ...

Our aim. When $T_2(\Lambda)$ is of finite representation type?

Gorenstein-Projective modules

M. Auslander and M. Bridger (1969). A Λ-module M in $\mathrm{mod}(\Lambda)$, is called Gorenstein projective if it is isomorphic to a syzygy of a totally acyclic complex $P^\bullet$ of finitely generated projective modules.

Recall that a complex

$$P^\bullet : \cdots \to P^{-1} \xrightarrow{d^{-1}} P^0 \xrightarrow{d^0} P^1 \to \cdots$$

of finitely generated projective Λ-modules is called totally acyclic if it is acyclic and the induced Hom complex $\mathrm{Hom}_\Lambda(P^\bullet, \Lambda)$ is also acyclic.

Finite Cohen-Macaulay type

$\mathscr{G}(\Lambda) =$ the full subcategory of $\mathrm{mod}(\Lambda)$ consisting of all Gorenstein projective modules.

A. Beligiannis (2011): An Artin algebra Λ is called of **finite CM-type** if there are only finitely many isomorphism classes of indecomposable Gorenstein projective modules.

Let $\mathscr{G}(\Lambda) = \mathrm{add}(G)$. Then $\mathrm{End}_\Lambda(G)^{\mathrm{op}}$, denoted by $\mathrm{Aus}(\mathscr{G}(\Lambda))$, is called the Cohen-Macaulay Auslander algebra of Λ.

If Λ is self-injective, then $\mathscr{G}(\Lambda) = \mathrm{mod}(\Lambda)$. Hence a self-injective algebra Λ is of finite CM-type if and only if it is of finite representation type.

Example

Let $\Lambda = k[x]/(x^n)$, where k is an algebraically closed field. Note that Λ is a representation-finite self-injective algebra.

Y. Drozd, and V. Mazorchuk (2006): Auslander algebra of Λ is of finite representation type if and only if $n \leq 3$.

So $T_2(\Lambda)$ is of finite representation type if and only if $n \leq 3$.

Z.-W. Li and P. Zhang (2010): If $n = 4$ or $n = 5$, then $T_2(\Lambda)$ is an algebra of finite CM-type.

Therefore $T_2(k[x]/(x^4))$ is an algebra of infinite representation type but of finite CM-type.

Functor category (I learned from Helmut)

Consider the collection $\mathscr{L}(\mathscr{G}(\Lambda))$ of all left exact sequences

$$G : G_2 \overset{f_2}{\hookrightarrow} G_1 \overset{f_1}{\to} G_0$$

in $\mathrm{mod}(\Lambda)$ such that $G_0, G_1, G_2 \in \mathscr{G}(\Lambda)$. By applying the Yoneda functor to G, we get the exact sequence

$$0 \longrightarrow (-, G_2) \xrightarrow{(-,f_2)} (-, G_1) \xrightarrow{(-,f_1)} (-, G_0) \longrightarrow F_G \longrightarrow 0$$

in $\mathrm{mod}(\mathscr{G}(\Lambda))$. Define

$$\mathscr{F} : \mathscr{L}(\mathscr{G}(\Lambda)) \to \mathrm{mod}(\mathscr{G}(\Lambda))$$

by $\mathscr{F}(G) := F_G$.

Functor category (I learned from Helmut)

There exists a functor

$$\vartheta : \mathrm{mod}(\mathscr{G}(\Lambda)) \longrightarrow \mathrm{mod}(\Lambda)$$

defined as follows. For $F \in \mathrm{mod}(\mathscr{G}(\Lambda))$ consider a projective presentation

$$(-, G_1) \xrightarrow{(-,f)} (-, G_0) \longrightarrow F \longrightarrow 0$$

in $\mathrm{mod}(\mathscr{G}(\Lambda))$, where G_1 and G_0 lie in $\mathscr{G}(\Lambda)$. Define

$$\vartheta(F) := \mathrm{Coker}(G_1 \longrightarrow G_0).$$

It follows from a result of JA, R. Hafezi and M.H. Keshavarz (2021) that, over 1-Gorenstein algebras, ϑ is an exact functor that admits left and fully faithful right adjoints.

Results

Recall

$$\vartheta : \mathrm{mod}(\mathscr{G}(\Lambda)) \longrightarrow \mathrm{mod}(\Lambda)$$

Theorem

Let Λ be a 1-Gorenstein algebra of finite CM-type. Then the subcategory $\vartheta^{-1}(\mathscr{G}(\Lambda))$ of $\mathrm{mod}(\mathscr{G}(\Lambda))$ is of finite representation type if and only if $T_3(\Lambda)$ is of finite CM-type.

Corollary

Let Λ be a self-injective algebra of finite representation type. Then $T_2(\Lambda)$ is of finite representation type if and only if $T_3(\Lambda)$ is of finite CM-type.

Results

Corollary

Let Λ be a 1-Gorenstein algebra of finite CM-type. If the Cohen-Macaulay Auslander algebra $\mathrm{Aus}(\mathscr{G}(\Lambda))$ of Λ is of finite representation type, then $T_3(\Lambda)$ is of finite CM-type.

Corollary

Let Λ and Λ' be finite dimensional algebras such that one of them is self-injective of finite representation type. If Λ and Λ' are derived equivalent, then the Auslander algebra of Λ is of finite representation type if and only if the Auslander algebra of Λ' is so.

References

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Thank you for your attention!