NP-completeness and higher τ-tilting theory

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Setting: $\mathrm{mod}\,\Lambda$

Λ is d-representation finite

$\mathcal{D}$ d-cluster tilting subcategory

- finitely many indecomposables
- $\mathrm{proj}\,\Lambda \subseteq \mathcal{D}$
- d-Auslander-Reiten sequences

$$0 \to \tau_d X \to Y_1 \to \dots \to Y_d \to X \to 0$$

Problem: Find all maximal τ_d-rigid pairs in $\mathcal{D}$

[J.-Jørgensen '20]

↳ (T, P) such that $T \in \mathcal{D}$, P projective, ("basic")

$\mathrm{Hom}(T, \tau_d T) = 0$

$\mathrm{Hom}(P, T) = 0$ + maximality

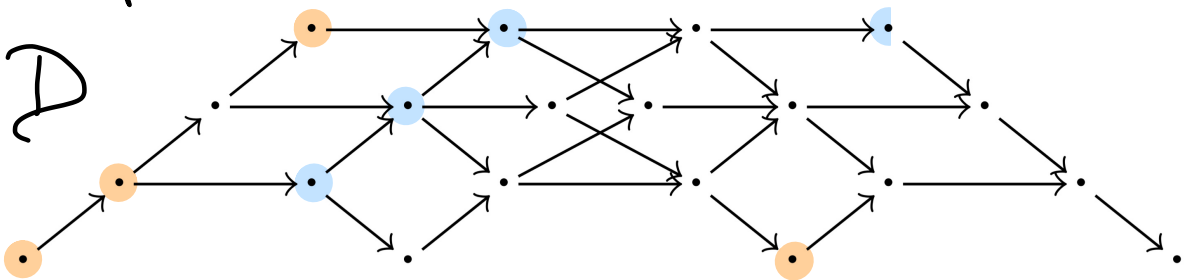
$d = 1$: support τ-tilting pairs

$|T| + |P| = |\Lambda|$

We'll restrict to $\Lambda = \mathbb{A}_n^d$ (higher Auslander Algebra)

Example: $\mathrm{mod}\,\mathbb{A}_4^2$

$\mathcal{D}$



(T, P) is a max. τ_2-rigid pair

Solution part 1 Let $T = \bigoplus T_i$, $P = \bigoplus P_j$ (indecomposable)

- Enough to check $\mathrm{Hom}(T_i, \tau_d T_k) = 0 \quad \forall i,k$
  $\mathrm{Hom}(P_j, T_i) = 0 \quad \forall i,j$
  and no more indecomposable summands can be added.

- Use a combinatorial model for $\mathcal{D} \subseteq \mathrm{mod}\, \mathbb{A}_n^d$
  [Oppermann-Thomas '12, Jasso-Külshammer '19]

- Plug it all into a computer....

... and realize it's still quite hard

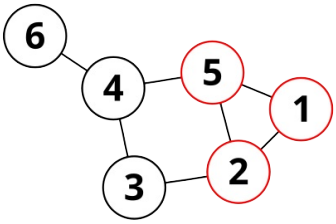
Abstraction: Consider a graph Γ (T, P)

vertices: $\mathrm{ind}\,\mathcal{D} \amalg \mathrm{ind\,proj}\,\Lambda$

edges: $M_i - M_j$ for $M_i, M_j \in \mathrm{ind}\,\mathcal{D}$
if $\mathrm{Hom}(M_i, \tau_d M_j) = 0 = \mathrm{Hom}(M_j, \tau_d M_i)$

$P_i - P_j$ $\forall P_i, P_j \in \mathrm{ind\,proj}\,\Lambda$

$M_i - P_j$ if $\mathrm{Hom}(P_j, M_i) = 0$



Look for maximal fully connected subgraphs

This is called the maximal clique problem and is studied in C.S. It is NP-hard

It is solved by the Bron-Kerbosch algorithm (1973) which has very good implementations

Results

	# max τ_d-rigid pairs for $\mathbb{A}_4^d$		
d	$\lvert T\rvert+\lvert P\rvert < \lvert\mathbb{A}_4^d\rvert$	$\lvert T\rvert+\lvert P\rvert = \lvert\mathbb{A}_4^d\rvert$	$\lvert T\rvert+\lvert P\rvert > \lvert\mathbb{A}_4^d\rvert$
1	0	42	0
2	0	357	0
3	23 100	3 278	0
4	29 478 592	35 798	0

Next questions

- Is $\lvert T\rvert+\lvert P\rvert \leq \lvert\Lambda\rvert$?
- Is there a pattern to the number of pairs?
- Can we use what we know about (higher) τ-tilting theory to understand the maximal clique problem?

Thanks for your attention!